MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

VOL. LXV.

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WITH MANY

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IN ADDITION TO THOSE

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EDITED BY

W. J. C. MILLER, B.A.,

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Solved Questions.

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MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

3882. (Professor TOWNSEND, F.R.S.)—The circumscribed and inscribed circles of a variable triangle, plane or spherical, being supposed both fixed, show that, throughout the deformation of the triangle, velocity of A : velocity of B : velocity of C = $\cot \frac{1}{2}A : \cot \frac{1}{2}B : \cot \frac{1}{2}C$, in either case; and hence that angular velocity of a : angular velocity of b : angular velocity of $c = a : b : c$ for the plane, and $= \tan \frac{1}{2}a : \tan \frac{1}{2}b : \tan \frac{1}{2}c$ for the spherical triangle.

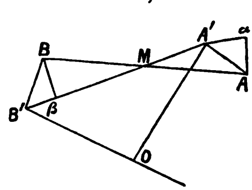
Solution by Profs. RAMACHANDRA ROW, KRISHMACHANDRA DE, *and others.*

Let AB, A′B′ be two consecutive positions of the side c. Since they are both tangents to the in-circle, they intersect at a point on the in-circle, say M. Draw $A\alpha$, $B\beta$ perpendicular to A′B′.

Then the velocity of A is proportional to element AA′, and the angular velocity of c to angle M.

Draw A′O perpendicular to AA′, and B′O to BB′; then O is circumcentre.

It can easily be shown to be



(This figure applies for spherical triangles also.)

$$OA'B' = s - c;$$

$$\therefore \quad AA'\alpha = \tfrac{1}{2}\pi - s + c = BB'\beta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

Plane Triangle.

$$\frac{AA'}{BB'} = \frac{A\alpha}{B\beta} = \frac{MA}{MB} = \frac{r \cot \frac{1}{2}A}{r \cot \frac{1}{2}B}; \quad \therefore \quad \frac{\text{velocity of A}}{\text{velocity of B}} = \frac{\cot \frac{1}{2}A}{\cot \frac{1}{2}B}.$$

$$\sin M = \frac{A\alpha}{MA} = \frac{AA' \sin(s-c)}{MA} = \frac{AA'}{r \cot \frac{1}{2}A} \sin C, \quad \because \quad s = \tfrac{1}{2}\pi;$$

and $\because$ $AA' \propto \cot \frac{1}{2}A$ and r is given,

$$\sin M \propto \sin C \propto c.$$

Spherical Triangle.

$$\frac{\sin AA'}{\sin BB'} = \frac{\sin A\alpha}{\sin B\beta} = \frac{\sin MA}{\sin MB} = \frac{\tan r \cot \frac{1}{2}A}{\tan r \cot \frac{1}{2}B}.$$

In the limit, $RAA' = AA'$ in both plane and spherical triangles.

$$\sin M = \frac{\sin A\alpha}{\sin MA} = \frac{\sin AA' \cos s-c}{\tan r \cot \frac{1}{2}A};$$

and $\because$ $\sin A'A' \propto \cot \frac{1}{2}A$ and r is given,

$$\sin M \propto \cos s-c \propto \tan \tfrac{1}{2}c \left(-\frac{\cos s-A \cos s-B \cos s-c}{\cos s}\right)^{\frac{1}{2}} \propto \tan \tfrac{1}{2}c.$$

In the limit, $\sin M = M$.

$\therefore$ angular velocity of $c \propto c$, angular velocity of $c \propto \tan \frac{1}{2}c$.

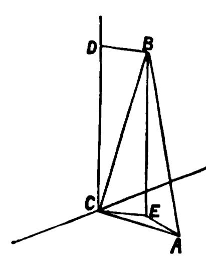
The constant of variation in both cases is not *a constant*, but a symmetrical function of the angles and sides.

12950. (A. S. Eve, M.A.)—A rod rests on rough ground against a rough vertical wall, and it is in a vertical plane inclined at an angle θ to the horizon. The base remains fixed, but the upper end is displaced sideways until slipping occurs. If the projection of the rod on the wall turns through an angle α, prove that the coefficient of friction between the rod and wall equals $\tan\alpha \tan\theta$.

Solution by H. W. Curjel, M.A.; Prof. Radhakrishnan; *and others.*

Let AB be the position of the rod when just about to slip, and AC be the perpendicular from A on the wall, and CD the vertical line through C. Then $\angle BAC = \theta$ and $\angle BCD = \alpha$.

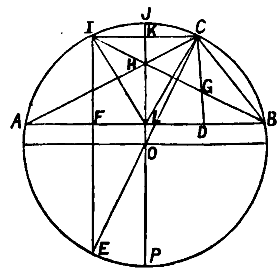
Draw BD in the plane of the wall at right angles to AB. Complete the parallelogram BDCE. Then the resultant reaction at B acts in the vertical plane through AB, and its projection on the wall is along BD. Hence EA, EC, CA clearly represent the resultant reaction, friction, and normal reaction at B. Thus we find



$$\mu = \frac{F}{R} = \frac{BD}{AC} = \tan\theta \,.\, \tan\alpha.$$

12894. (Cecil Ewing.)—Given the diameter of the circumscribed circle, the sum of the base and the perpendicular, and one base angle double the other, to construct the triangle.

Solution by M. BRIERLEY, Professor CHAKRIVARTI, *and others.*



Let ABC be the required triangle, and CE the diameter of the circum-circle; then, since $\angle ABC = 2CAB$, if BI be drawn bisecting the angle ABC, it will be perpendicular to CE. Draw the diameter JKHLOP, perpendicular to, and intersecting the base AB in L; join C, I; C, L; and LI, and draw the perpendicular CD, cutting BI in G; also let EI be drawn, cutting the base in F. Then BGC, CHI are evidently similar and equal triangles;

$$\therefore \quad BG = GC = HC = GH,$$

and $$BG : BH = BD : BL = BD = DL,$$

and $$2DG = HL = \tfrac{2}{3}CD - CLI,$$

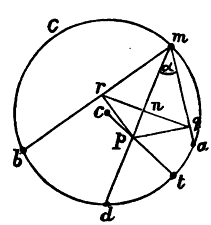
or BCL is an equilateral triangle, and

$$CL = CI = CB = BL, \quad \text{and} \therefore \; KL = CD = \tfrac{1}{4}\sqrt{3}\,AB;$$

hence AB is given, and the construction manifest.

12943. (Professor MANNHEIM.)—On donne une circonférence de cercle C et deux points *a* et *b* sur cette courbe. On mène les droites *am*, *bm* qui aboutissent au point *m* de C. On décrit une circonférence de cercle tangente à C, et à ces droites on prend la corde de contact de ce cercle et de ces droites. Démontrer que lorsque *m* décrit C cette corde de contact reste tangente à une circonférence de cercle.

Solution by R. F. DAVIS, M.A.; Professor A. DROZ-FARNY; *and others.*



Let p be the centre of the circle touching ma, mb in q, r respectively, and also touching C in t. Then, if d be the middle point of the arc ab of C, md passes through p and bisects qr at right angles in n. Then (α being $\angle amd$), from

$$ca^2 - cp^2 = pm \,.\, pd, \quad cp + pm \sin\alpha = ca,$$

we have $pd + pm \sin^2\alpha = 2ca \sin\alpha$,

$$pd + pn \; (= dn) = da \; (= db);$$

hence n is the in-centre of the triangle mab, and qr touches at n the fixed circle (centre d, radius da) which is the locus of n. Let p be the centre of the circle touching ma, mb in q, r respectively, and also touching C in t. Then, if d be the middle point of the arc ab of C, md passes through p and bisects qr at right angles in n.

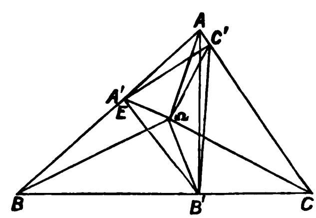
[If we compare this with MCCLELLAND's *Geometry of the Circle* ("Mannheim's Theorem," pp. 10, 11), we shall find this proof much simpler.]

12934. (Professor Sanjána.)—Through the vertices of a triangle ABC are drawn the lines AB_1, BC_1, CA_1 to meet the opposite sides and to make angles such that

$$\cot BAB_1 = \cot A + \cot C, \quad \cot CBC_1 = \cot B + \cot A,$$
$$\cot ACA_1 = \cot C + \cot B.$$

Prove that (1) the triangle $A_1B_1C_1$ is similar to ABC, the ratio of similarity being $\tan \omega$; (2) the circles drawn round AC_1A_1, BA_1B_1, CB_1C_1 meet in one point; and (3) this point is a centre of similitude of the triangles, the corresponding vertices being separated by a right angle about it.

Solution by Professors A. Droz-Farny, Mukhopadhyay, *and others.*



Ω étant le premier point de Brocard, on sait que, si le faisceau Ω (A, B, C) tourne autour de Ω dans le sens de rotation ABC, les rayons ΩA, ΩB, ΩC coupe respectivement les côtés AB, BC, CA en les points A_1, B_1, C_1 tels que le triangle $A_1B_1C_1$ est semblable à ABC. Le rapport de similitude égale $\Omega A_1 : \Omega A$; d'après la théorie des figures semblables les circonférences AC_1A_1, BA_1B_1, CB_1C_1 se coupent évidemment en Ω, centre de similitude des deux triangles. Faisons tourner le faisceau de 90°, comme angle $B\Omega B_1 = 90°$ et que angle $\Omega BC = \omega$ on a pour le rapport de similitude $\Omega B_1 : \Omega B = \tan \omega$. Abaissons B_1E perpendiculaire sur AB. On sait que

$$B\Omega = c \sin \omega / \sin B; \quad \text{donc} \quad BB' = \cot \omega / \sin B.$$

Il en résulte $\quad AE = c - \cot \omega \tan B, \quad EB' = \cot \omega.$

Et par conséquent $\cot BAB_1 = (1 - \tan \omega \tan B)/\tan \omega = \cot \omega - \cot B,$

$$\cot BAB_1 = \cot A + \cot C.$$

Comme $BB' = \cot \omega / \sin B$ et $A'B' = \cot \omega$, il en résulte que $A'B'/BB' = \sin B$ et que $B'A'$ est perpendiculaire sur AB. Le point E de la démonstration coincide avec A′. Le triangle A′B′C′ a ses côtés perpendiculaires sur les côtés homologues de ABC.

3854. (Professor Sir R. S. Ball.)—From any point perpendiculars are drawn to the generators of the surface $z(x^2 + y^2) - 2mxy = 0$. Show that the feet of the perpendiculars lie upon a plane ellipse.

Solution by H. W. Curjel, M.A.; Professor Sarkar; *and others.*

The generators are given by

$$\frac{x}{\cos \theta} = \frac{y}{\sin \theta + 1} = \frac{z - m \cos \theta}{0},$$

and, where the perpendicular from (a, b, c) meets this,

$$(x-a)\cos\theta + (y-b)(1+\sin\theta) = 0.$$

Hence the locus lies on the cylinder $x(x-a) + y(y-b) = 0$.

Solving the equations for x and y, we get

$$bx = \frac{ab(1-\sin\theta) + b^2\cos\theta}{2}, \quad ay = \frac{ab(1+\sin\theta) + a^2\cos\theta}{2};$$

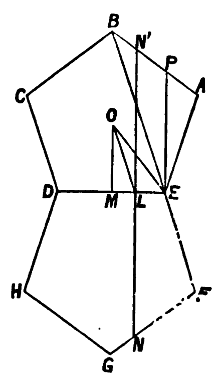
$$\therefore \quad bx + ay = \frac{(a^2+b^2)\cos\theta + 2ab}{2} = \frac{(a^2+b^2)z}{2m} + ab;$$

$\therefore$ locus lies on the plane $2m(bx+ay) = (a^2+b^2)z + 2abm$.

$\therefore$ the locus is a plane ellipse.

12951. (V. J. Bouton, B.Sc., F.R.A.S.)—Two regular pentagons ABCDE, DEFGH are drawn in a plane, one side DE being common. Through the centre O of the first pentagon is drawn a straight line OL parallel to the side CD, cutting DE in L. Through L is drawn NLN′ perpendicular to DE; find the ratio in which N cuts FG, or N′ cuts AB.

Solution by I. Arnold; H. W. Curjel, M.A.; *and others.*



Draw OM perpendicular to DE. Draw EP perpendicular to DE cutting BA in P.

Now OL bisects $\angle$ MOE and EP bisects $\angle$ BEA.

$$\therefore \quad \frac{BN'}{N'P} = \frac{ML}{LE} = \frac{MO}{OE} = \cos 36^\circ = \frac{\sqrt{5}+1}{4},$$

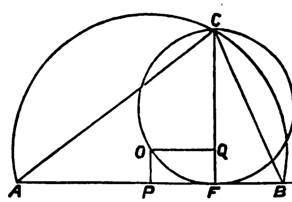
and
$$\frac{BP}{PA} = \frac{BE}{EA} = \frac{2}{\sqrt{5}-1};$$

$$\therefore \quad \frac{BN'}{BA} = \frac{BN'}{BP} \cdot \frac{BP}{BA} = \frac{1}{\sqrt{5}} \times \frac{2}{\sqrt{5}+1};$$

$$\therefore \quad \frac{GN}{NF} = \frac{BN'}{N'A} = \frac{2}{3+\sqrt{5}} = \frac{3-\sqrt{5}}{2}.$$

12889. (Professor Sanjána.)—In a triangle, AD, BE, CF are the altitudes, and O is the centre of the circumcircle; prove that, if $\sin^2(B-A) = \cos C \cos(B-A)$, then the circle on CF as diameter passes through O. Find the angles of the triangle when the circles on BE and CF both pass through O; and prove that it is not possible for the circles on AD, BE, CF to co-intersect in O.

Solution by H. W. CURJEL, M.A.; Professor CHAKRIVARTI; *and others.*



Let P be the mid-point of AB, and draw OQ perpendicular to CF; then $R\cos C = QF$,

$R\sin(B-A) = OQ$,

for $\angle OCF = B-A$.

But $\cos C\cos(B-A) = \sin^2(B-A)$;

therefore $QF/OQ = \tan(B-A)$;

therefore $\angle FOQ = (B-A)$;

therefore $\angle COF$ is a right angle; therefore circle on CF as diameter passes through O.

Also if O is on the circles on CF, BE as diameters

clearly $\sin^2(B-A) = \cos C\cos(B-A)$, $\sin^2(A-C) = \cos B\cos(A-C)$;

$\therefore\ \sin^2(B-A) = \sin^2 A-\cos^2 B$, and $\sin^2(A-C) = \sin^2 A-\cos^2 C$;

therefore $\sin(B-C)\sin(B+C-2A) = \sin(B-C)\sin(B+C)$;

therefore $B = C$ or $B+C-A = \frac{1}{2}\pi$, *i.e.*, $A = \frac{1}{4}\pi$.

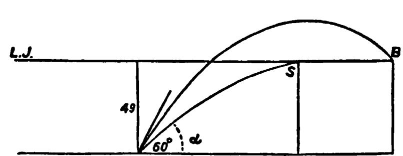
If $B = C$, we get $B = C = \frac{1}{4}\pi$ or $\sin^{-1}\frac{1}{4}[6+2(17)^{\frac{1}{2}}]^{\frac{1}{2}}$.

If $A = \frac{1}{4}\pi$, the Δ is a right-angled isosceles Δ.

Hence the Δ is a right-angled isosceles Δ, in which case the circles on AE, BE, CF co-intersect in O, or $B = C = \sin^{-1}\frac{1}{4}[6+2(17)^{\frac{1}{2}}]^{\frac{1}{2}}$.

12834. (Rev. T. ROACH, M.A.)—ROBIN HOOD was standing in a hollow, and shot an arrow at an angle of 60°, and with a velocity of $3\sqrt{3g}$ feet per second, at a buck standing $4g$ feet above his own level. LITTLE JOHN, who was standing on the same level as the buck, but $4\sqrt{3g}$ feet behind ROBIN HOOD, shot an arrow, 4 seconds after ROBIN HOOD's, with a velocity $4\sqrt{3g}$ feet per second, directly towards the buck. ROBIN HOOD, hearing LITTLE JOHN prepare to shoot, and fearing the honours of the shot would be divided if they hit the buck at the same instant, shot an arrow 1 second after LITTLE JOHN's, which pierced the second arrow on its course 1 second before it would have struck the buck, and spoilt the shot. Show that ROBIN HOOD was correct as to the time of arrival of the second shot, and find the velocity and direction of the third.

Solution by Rev. J. L. KITCHIN, M.A.; Prof. RADHAKRISHNAN; *and others.*



The path of the arrow is

$$y = x\sqrt{3}-\frac{2gx^2}{27g^2},$$

when $x = 12\sqrt{3g}$

= flight horizontally to O;

$\therefore\ 16\sqrt{3g}$ = LITTLE JOHN's arrow flight, time 4 secs.

Robin Hood's horizontal velocity $= \frac{1}{2}\sqrt{3g}$; $\therefore$ time of flight $= 8$ secs.; therefore both arrows would strike the buck at the same instant.

Robin Hood's second shot is made when Little John's arrow is just passing over his head; time of flight, 2 secs.; x and y coordinates of meeting of arrows, $8\sqrt{3}g$, $4g$.

Let v = velocity, α = angle of elevation; then

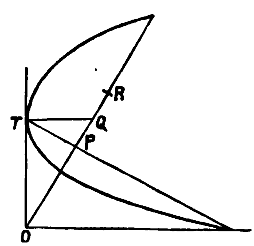
$$2v\cos\alpha = 8\sqrt{3}g; \quad \therefore \quad v\cos\alpha = 4\sqrt{3}g.$$

$$\text{Path} = 4g = 8\sqrt{3}g\tan\alpha - \frac{g\,(8\sqrt{3}g)^2}{32\,(\sqrt{3}g)^2};$$

$$\therefore \quad \tan\alpha = \tfrac{1}{4}\sqrt{3}; \quad \therefore \quad \cos\alpha = 4/\sqrt{19}; \quad \therefore \quad v = \sqrt{57}g.$$

12921. (A. S. Eve, M.A.)—A straight line passes through a fixed point O, and meets two fixed straight lines in P, Q. If OR is a third proportional to OP, OQ, find the locus of R.

Solution by Rev. E. S. Longhurst, B.A.; H. W. Curjel, M.A.; *and others.*



This proposition is the converse of Question 12893, and the locus is a parabola.

For, in 12893, P moves along the polar, Q moves along the parallel through T to the axis, and OP, OR = OQ².

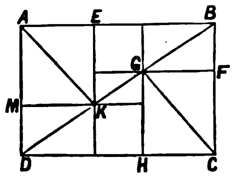
But R is the locus of mid-points of chords through a fixed point to a parabola.

Hence the locus of R is a parabola.

[The solution by polar coordinates is very simple.]

12905. (Professor Shields.)—A gentleman owned and lived in the centre, R, of a rectangular tract of land whose diagonal, D, was 350 rods, dividing the tract into two equal right-angled triangles, in each of which is inscribed the largest square field F and F possible; the north and south boundary line of the two square fields being extended and joined formed a little rectangular lot R, in the centre around the residence. The difference in the area of the entire rectangular tract and the sum of the areas of the two square fields F, F, is $187\frac{1}{2}$ acres. Give the dimensions and area of the entire tract, and one square field F.

Solution by Professors Zerr, Radhakrishnan, *and others.*



Draw CG, AK making angles of 45° with the sides; then will HCFG, MKEA be the squares.

Let $AB = a$, $BC = b$, $FC = x$.

Then $a : b = a - x : x$; $\therefore\ x = ab/(a+b)$;

$ab - 2x^2 = 187\frac{1}{2}$ acres $= 30{,}000$ sq. rods ... (1);

$a^2 + b^2 = 122{,}500$ .... (2);

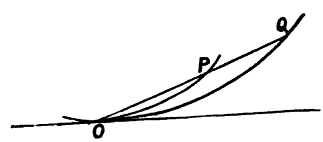
$\therefore\quad ab = 58{,}800$ rods $= 367\frac{1}{2}$ acres,

$a + b = 490$ rods, $a - b = 70$ rods;

$\therefore\quad a = 280,\quad b = 210,\quad x = 120,\quad x^2 = 14{,}400$ sq. rods $= 90$ acres.

3857. (Professor Whitworth).—Two curves touch one another, and both are on the same side of the common tangent. If, in the plane of the curves, this tangent revolves about the point of contact, or if it move parallel to itself, show that the prime ratio of the nascent chords in the former case is the duplicate of their prime ratio in the latter case.

Solution by Professors Bhattacharya, Ramachandra Row, *and others.*

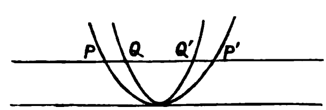


Take the tangent and normal as axes; the curves are in the limit parabolas.

Let the equations be

$2fy = ax^2 + \&c.,\quad 2f'y = a'x^2 + \&c.$

$$L_\theta = Lt\,\frac{OP}{OQ} = Lt\,\frac{2f\sin\theta \div a\cos^2\theta}{2f'\sin\theta \div a'\cos^2\theta} = fa'/f'a.$$



$$L_y = Lt\,\frac{PP'}{QQ'}$$

$$= Lt\,\frac{+\left(\frac{2fy}{a}\right)^{\frac12} - \left\{-\left(\frac{2fy}{a}\right)^{\frac12}\right\}}{\left(\frac{2f'y}{a'}\right)^{\frac12} - \left\{-\left(\frac{2f'y}{a}\right)^{\frac12}\right\}}$$

where y is the same,

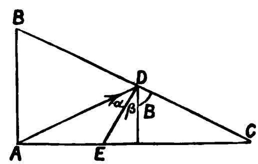
$$= (fa'/f'a)^{\frac12} = (L_\theta)^{\frac12}.$$

This is true in the limit for all curves, and true for parabolas always.

[This theorem requires some such limitation as that the mid-points of the nascent chords parallel to the common tangent should be coincident, as is clear when we consider the case of a curve and its reflexion with respect to a normal, in the case where the normal does not pass through the mid-point of the nascent chord parallel to the common tangent; for in the first case the nascent chords are not generally equal, while in the second they always are equal.]

12915. (Professor CALDERHEAD.)—Show that, if a body be projected from the angle A of a plane triangle ABC so as to strike the side CB at a point D, then, if its course after reflection at D be parallel to AB, $\tan DAB = (1+E)\cot B/(1-E)\cot^2 B$.

Solution by Rev. E. S. LONGHURST, B.A.; H. W. CURJEL, M.A.; *and others.*



Let α = angle of incidence,

β = angle of reflexion.

Then, by resolving parallel and perpendicular to CB, we have

$$\tan\alpha = e\tan\beta;$$

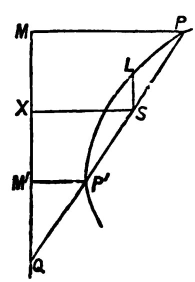
and $\tan DAB = \tan(\alpha+\beta)$

$$= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \frac{(1+e)\cot B}{1-e\cot^2 B};$$

$$\because\quad \beta+B = \tfrac{1}{2}\pi.$$

3883. (M. COLLINS, B.A.)—The harmonic mean between the segments of any chord of a conic section passing through its focus is constant and = the semi-parameter; required a demonstration true and general for all the three conics.

Solution by Professors RAMACHANDRA ROW, KRISHMACHANDRA DE, *and others.*



$$SP : SP' = PM : P'M' = QP : QP';$$

$$\therefore\quad QP : QP' = QP-QS : QS-QP'.$$

Therefore QP, QS, QP′ are in harmonic progression. But

$$QP : QS : QP' = PM : SX : P'M'$$

by similar triangles,

$$= SP : SL : SP'$$

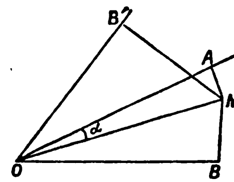
from the definition of the conic.

Therefore SP, SL, SP′ are in harmonic progression.

12858. (Professor MOREL.)—D'un point quelconque M du plan d'un angle BOB′, égal à 60°, on abaisse les perpendiculaires MB, MB′, MA sur les côtés OB, OB′ et sur la bissectrice de cet angle. Démontrer que

$$OB = MB'-MB.$$

Solution by Professor SANJÁNA, Rev. J. L. KITCHIN, *and others.*



$$MB' = OM \sin(30^\circ + \alpha),$$
$$MB = OM \sin(30^\circ - \alpha);$$
$$\therefore \quad MB' - MB = 2\,OM \cos 30^\circ \sin\alpha = \sqrt{3}\,MA,$$
$$\text{and } MB' + MB = 2\,OM \sin 30^\circ \cos\alpha = OA.$$

So also $\quad OB - OB' = AM.$

The result given is obviously erroneous.

12448. (Professor CATALAN.)—On satisfait à l'équation

$$(1) \quad x^2 + y^2 = z^2$$

en prenant:
$$x = \alpha^{m} - C_{2n,2}\,\alpha^{2n-2}\beta^2 - C_{2n,4}\,\alpha^{2n-4}\beta^4 - \dots,$$
$$y = C_{2n,1}\,\alpha^{2n-1}\beta - C_{2n,3}\,\alpha^{2n-3}\beta^3 + \dots,$$
$$z = (\alpha^2 + \beta^2)^n.$$

En particulier, $\quad x = \alpha^2 - \beta^2, \quad y = 2\alpha\beta, \quad z = \alpha^2 + \beta^2.$

Si l'on a trouvé une valeur de z, $z = c$, satisfaisant à l'équation (1), toutes les puissances, entières et positives, de c, sont aussi des valeurs de z.

Solution by H. J. WOODALL, A.R.C.S.; Professor LAMPE; *and others.*

We have $\quad (x^2 + y^2) = (x + y\iota)(x - y\iota) \quad \{\text{where } \iota = (-1)^{\frac{1}{2}}\}$
$$= (\alpha + \iota\beta)^{2n}(\alpha - \iota\beta)^{2n} = (\alpha^2 + \beta^2)^{2n} = z^2,$$
which proves the first part.

For the second part, if we can find x, y when $z = c$, we must be able to decompose c into the sum of two squares $= \alpha^2 + \beta^2$. Similarly we can find x, y when $z =$ any whole and positive power of c.

Example (1), $\quad x = \alpha^2 - \beta^2, \quad y = 2\alpha\beta, \quad z = \alpha^2 + \beta^2;$

(2) $\quad x = \alpha^4 - 6\alpha^2\beta^2 + \beta^4, \quad y = 4\alpha^3\beta - 4\alpha\beta^3, \quad z = (\alpha^2 + \beta^2)^2;$

and so on.

[CHRYSTAL proves (*Algebra*, Vol. II., p. 503) that the most general form possible is $\quad x = \lambda(m^2 - n^2), \; y = 2\lambda mn, \; z = \lambda(m^2 + n^2),$
to which we may easily reduce (2) above.

In a similar manner, if c be a value of z satisfying $x^2 - y^2 = z^2$, then any whole and positive power of c will also be a possible value of z.]

12678. (Professor IGNACIO BEYENS.)—Dans quel cas on vérifie en un quadrilatère que la somme des carrés des côtés est égale au double du carré d'une diagonale? On vérifie d'abord dans un carré mais aussi dans des autres cases.

Solution by H. W. Curjel, M.A.; Professor Chakrivarti; *and others.*

The condition that the sum of the squares on the sides of a quadrilateral should be equal to twice the square on a diagonal may evidently be put in the form: that the circle on that diagonal as diameter should pass through the ends of a diameter of the circle on the other diagonal as diameter; or that it should be possible to place the two diagonals and twice the distance between their middle points so as to form a right-angled triangle.

12097. (Professor Bernès.)—Sur les trois côtés d'un triangle ABC, on prend trois segments quelconques DD′, EE′, FF′. Démontrer que les axes radicaux des circonférences circonscrites à AEF, AE′F′; BFD, BF′D′; CDE, CD′E′ sont trois droites concourantes. Deux des segments étant donnés, déterminer la troisième par la condition que le point de concours soit commun aux six circonférences.

Solution by Professors A. Droz-Farny, Mukhopadhyay, *and others.*

Considérons les longueurs DD′ sur BC, EE′ sur AC et FF′ sur AB comme les segments homologues de trois figures semblables X_1, X_2, X_3.

Les circonférences AEF et AE′F′ se coupent en O_1, point double des figures X_2 et X_3. Or, d'après Casey, *A Sequel to Euclid*, p. 185: "In every system of three figures directly similar, the triangle formed by three homologous lines is in perspective with the triangle $O_1O_2O_3$ of similitude."

AO_1, BO_2, CO_3 se coupent donc en un même point.

Construisons d'abord le point O_1, intersection des circonférences AEF et AE′F′. La circonférence BFO_1 coupera BC en D et la circonférence $CE'O_1$ coupera BC en D′.

12805. (W. C. Stanham.)—If $f_1(\theta)' = i \log(\sec\theta + \tan\theta)$, where i denotes $(-1)^{\frac{1}{2}}$, and if $f_{r+1}(\theta) = f_1\{f_r(\theta)\}$, prove that

$$f_{2m}(\theta) = (-1)^m \theta \pm 2n\pi.$$

Solution by H. W. Curjel, M.A.; Professor Mukhopadhyay; *and others.*

$$f_1(\theta) = i\log(\sec\theta + \tan\theta) = i\cosh^{-1}\sec\theta;$$

$$\therefore\ f_2(\theta) = i\cosh^{-1}\sec(i\cosh^{-1}\sec\theta) = i\cosh^{-1}\frac{1}{\cos(i\cosh^{-1}\sec\theta)}$$

$$= i\cosh^{-1}\cos\theta = (-1)\{\pm(\theta \pm 2n\pi)\};$$

$$\therefore\ f_{2m}(\theta) = \pm(\theta \pm 2n\pi).$$

[This form results from the assumption that the two values (i and $-i$) of $(-1)^{\frac{1}{2}}$ may be used indifferently, whenever the operator f_1 is applied; the Proposer's from the assumption that one value only may be used.]

12749. (Professor HUDSON, M.A.)—If a square number end in 6, prove that the previous figure is odd; if it end in 9, prove that the previous figure is even.

Solution by Professors A. DROZ-FARNY, RADHAKRISHNAN, *and others.*

Le théorème proposé trouve immédiatement sa solution par les égalités

$$(10a+4)^2 = 10\,(10a^2+8a+1)+6, \quad (10a+6)^2 = 10\,(10a^2+12a+3)+6;$$
$$(10a+3)^2 = 10\,(10a^2+6a)+9, \quad (10a+7)^2 = 10\,(10a^2+14a+4)+9.$$

Dans le premier cas le chiffre qui précède le 6 est impair; dans le deuxième cas le chiffre qui précède le 9 est pair.

9819. (W. J. C. SHARP, M.A.)—If the sides AB and AC of a spherical triangle ABC be divided in F and E respectively, so that sin AF : sin BF :: sin AE : sin CE, the great circle FE will cut the great circle BC in a point Q such that BQ + CQ = π, and the great circles through all such divisions meet in the same points, and conversely.

Solution by H. J. WOODALL, A.R.C.S.; Prof. BASU; *and others.*

Join AQ; then we have sin BQ : sin BF = sin BFQ : sin BQF,

sin AF : sin AQ = sin AQF : sin AFQ,

sin BF : sin AF = sin EC : sin AE;

multiplying, sin BQ : sin AQ = sin AQF $\times$ sin EC : sin BQF $\times$ sin AE.

So we can find sin CQ : sin AQ = sin AQE $\times$ sin EC : sin CQE $\times$ sin AE.

Therefore sin BQ = sin CQ; $\therefore$ BQ + CQ = π. This is independent of the ratio sin EC : sin AE; therefore all great circles through all such divisions meet in the same points, and conversely.

12093. (Professor ZERR.)—Find the volume common to the solids whose surfaces are given, where $a > b > c$, by

$$(x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}} + (z/c)^{\frac{2}{3}} = 1, \qquad x^{\frac{2}{3}} + y^{\frac{2}{3}} = b^{\frac{2}{3}};$$
$$(x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}} + (z/c)^{\frac{2}{3}} = 1, \quad x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = b^{\frac{2}{3}}.$$

Solution by the PROPOSER, Professor MUKHOPADHYAY, *and others.*

$$(1) \quad V = 8\iiint dx\,dy\,dz = 8\iint z\,dy\,dx.$$

The limits of x are 0 and $(b^{\frac{2}{3}} - y^{\frac{2}{3}})^{\frac{3}{2}} = x_1$; and of y, 0 and b.

$$\therefore \quad \mathrm{V} = \frac{8c}{ab}\int_0^b\int_0^{x_1}(a^{\frac{2}{3}}b^{\frac{2}{3}} - b^{\frac{2}{3}}x^{\frac{2}{3}} - a^{\frac{2}{3}}y^{\frac{2}{3}})^{\frac{3}{2}}\,dy\,dx$$

$$= \left\{\frac{4c}{ab}(a^{\frac{2}{3}} - b^{\frac{2}{3}})^{\frac{3}{2}} + \frac{3c\,(2b^{\frac{2}{3}} - a^{\frac{2}{3}})\,(a^{\frac{2}{3}} - b^{\frac{2}{3}})^{\frac{1}{2}}}{2a^{\frac{2}{3}}b^{\frac{2}{3}}} + \frac{3ac}{2b^2}\sin^{-1}\frac{b}{a}\right\}\int_0^b(b^{\frac{2}{3}} - y^{\frac{2}{3}})^3\,dy$$

$$= \frac{64b^2c\,(a^{\frac{2}{3}} - b^{\frac{2}{3}})^{\frac{3}{2}}}{105a} + \frac{8b^{\frac{4}{3}}c\,(2b^{\frac{2}{3}} - a^{\frac{2}{3}})\,(a^{\frac{2}{3}} - b^{\frac{2}{3}})^{\frac{1}{2}}}{35a^{\frac{2}{3}}} + \frac{8abc}{35}\sin^{-1}\frac{b}{a}.$$

(2) The limits of x are 0 and $\dfrac{a}{b}\left\{\dfrac{(b^{\frac{2}{5}} - c^{\frac{2}{5}})\,(b^{\frac{2}{5}} - y^{\frac{2}{5}})}{a^{\frac{2}{5}} - c^{\frac{2}{5}}}\right\}^{\frac{5}{2}} = x_1$, and also $(b^{\frac{2}{5}} - y^{\frac{2}{5}})^{\frac{5}{2}} = x_2$ and x_1; of y, 0 and b.

$$\therefore \quad \mathrm{V} = 8\int_0^b\int_0^{x_1}\frac{c}{ab}\left\{a^{\frac{2}{5}}(b^{\frac{2}{5}} - y^{\frac{2}{5}}) - b^{\frac{2}{5}}x^{\frac{2}{5}}\right\}^{\frac{5}{2}}dy\,dx + 8\int_0^b\int_0^{x_2}(b^{\frac{2}{5}} - y^{\frac{2}{5}} - x^{\frac{2}{5}})^{\frac{5}{2}}\,dy\,dx$$

$$= \left[\frac{15\pi}{64} + \frac{15ac}{32b^2}\sin^{-1}\left\{\frac{b^{\frac{2}{5}} - c^{\frac{2}{5}}}{a^{\frac{2}{5}} - c^{\frac{2}{5}}}\right\}^{\frac{1}{2}} - \frac{15}{32}\sin^{-1}\frac{a}{b}\left\{\frac{b^{\frac{2}{5}} - c^{\frac{2}{5}}}{a^{\frac{2}{5}} - c^{\frac{2}{5}}}\right\}^{\frac{1}{2}}\right.$$

$$- \frac{5ac}{2b^2}\left\{\frac{(a^{\frac{2}{5}} - b^{\frac{2}{5}})^{\frac{3}{2}}\,(b^{\frac{2}{5}} - c^{\frac{2}{5}})^{\frac{7}{2}}}{c^{\frac{2}{5}}\,(a \;- c^{\frac{2}{5}})^4}\right\} - \frac{5ac}{4b^2}\left\{\frac{(a^{\frac{2}{5}} - b^{\frac{2}{5}})^{\frac{1}{2}}\,(b^{\frac{2}{5}} - c^{\frac{2}{5}})^{\frac{7}{2}}\,(b^{\frac{2}{5}} + c^{\frac{2}{5}})}{c^{\frac{4}{5}}\,(a^{\frac{2}{5}} - c^{\frac{2}{5}})^3}\right\}$$

$$+ \frac{5ac}{16b^2}\left\{\frac{(b^{\frac{6}{5}} - c^{\frac{4}{5}}a^{\frac{2}{5}})\,(a^{\frac{2}{5}} - b^{\frac{2}{5}})^{\frac{1}{2}}\,(b^{\frac{2}{5}} - c^{\frac{2}{5}})^{\frac{5}{2}}}{c^{\frac{4}{5}}\,a^{\frac{2}{5}}\,(a^{\frac{2}{5}} - c^{\frac{2}{5}})^2}\right\}$$

$$\left. + \frac{15ac}{32b^2}\left\{\frac{(b^{\frac{6}{5}} - a^{\frac{4}{5}}c^{\frac{4}{5}})\,(a^{\frac{2}{5}} - b^{\frac{2}{5}})^{\frac{1}{2}}\,(b^{\frac{2}{5}} - c^{\frac{2}{5}})^{\frac{3}{2}}}{a^{\frac{4}{5}}\,c^{\frac{4}{5}}\,(a^{\frac{2}{5}} - c^{\frac{2}{5}})}\right\}\right]\int_0^b(b^{\frac{2}{5}} - y^{\frac{2}{5}})^5\,dy.$$

$$\mathrm{V} = \mathrm{A}\int_0^b(b^{\frac{2}{5}} - y^{\frac{2}{5}})^5\,dy \quad \text{suppose} \quad = \frac{256b^3}{9009}\,\mathrm{A}.$$

11929. (Professor MATZ, M.A.)—Three points are taken at random in the surface of a given elliptic quadrant; find (1) the mean area of all the triangles that can be formed by joining the random points with straight lines; and (2), by making the elliptic quadrant equiaxial, obtain therefore the result to Question 6285.

Solution by Professors ZERR, BHATTACHARYA, *and others.*

A solution to the first part of this Question is found on pp. 188–189 of Vol. LV., where the result is (1) $\Delta = ab/\pi\,(\frac{35}{12} + \frac{16}{3}\pi + \frac{131}{3}\pi^2)$;
(2) if $a = b = r$, $\Delta = r^2/\pi\,(\frac{35}{12} + \frac{16}{3}\pi - \frac{131}{3}\pi^2$.

11924. (Professor ZERR.)—From an unknown number of balls, each equally likely to be any of n colours, $a_1 + a_2 + \ldots + a_n$ balls are drawn, and turn out a_1 of the first colour, a_2 of the second colour, ..., a_n of the

n^{th} colour. If $b_1+b_2+\ldots+b_n$ more balls are drawn, find the probability that b_1 are of the first colour, b_2 of the second colour, ..., b_n of the n^{th} colour.

Solution by the PROPOSER.

Suppose the balls arranged along a straight line of length unity on n different portions of the line. Call the first portion x_{n-1}, the sum of the first and second portions, x_{n-2}, the sum of the first three portions, x_{n-3}, ..., the sum of the first $(n-1)$ portions, x_1; then we get for the required chance the following integral

$$p = \frac{(b_1+b_2+b_3+\ldots+b_n)!\int_0^1\int_0^{x_1}\int_0^{x_2}\ldots\int_0^{x_{n-2}}(1-x_1)^{a_1+b_1}(x_1-x_2)^{a_2+b_2} \times (x_2-x_3)^{a_3+b_3}\ldots x_{n-1}^{a_n+b_n}\,dx_1\,dx_2\,dx_3\ldots dx_{n-1}}{b_1!\,b_2!\,b_3!\ldots b_n!\int_0^1\int_0^{x_1}\int_0^{x_2}\ldots\int_0^{x_{n-2}}(1-x_1)^{a_1}(x_1-x_2)^{a_2}(x_2-x_3)^{a_3}\ldots \ldots x_{n-1}^{a_n}\,dx_1\,dx_2\,dx_3\ldots dx_{n-1}}$$

$$= \frac{(b_1+b_2+b_3+\ldots+b_n)!\,(a_1+b_1)!\,(a_2+b_2)!\,(a_3+b_3)!\ldots(a_n+b_n)! \times (a_1+a_2+a_3+\ldots+a_n+n-1)!}{b_1!\,b_2!\,b_3\ldots b_n!\,a_1!\,a_2!\,a_3!\ldots a_n! \times (a_1+a_2+a_3+\ldots+a_n+b_1+b_2+b_3+\ldots+b_n+n-1)},$$

when $n=3$,

$$p = \frac{(b_1+b_2+b_3)!\,(a_1+b_1)!\,(a_2+b_2)!\,(a_3+b_3)!\,(a_1+a_2+a_3+2)!}{b_1!\,b_2!\,b_3!\,a_1!\,a_2!\,a_3!\,(a_1+a_2+a_3+b_1+b_2+b_3+2)!},$$

when $n=3$, $a_1=5$, $a_2=3$, $a_3=2$, $b_1=b_2=b_3=1$; $p=72/455$.

3934. (Professor HUDSON, M.A.)—If the happiness which a person derives from his property increase with the property but at a diminishing rate, prove that, if a certain amount of property is to be divided among a certain number of persons, the greatest happiness will be secured by giving them equal shares. What will be the case if the happiness increase with the property (1) uniformly, (2) at an increasing rate?

Solution by Professors RAMACHANDRA ROW, RADHAKRISHNAN, *and others.*

Let h represent happiness and p property.

By question, $dh/dp=+$, $d^2h/dp^2=-$. Let $h=f(p)$.

$u\equiv h_1+h_2+\ldots+h_n=f(p_1)+f(p_2)+\ldots+f(p_n)$ is to be a maximum subject to the condition $p_1+p_2+\ldots+p_n=\text{P}$, a constant.

$$du = 0 = f'(p_1)\,dp_1+f'(p_2)\,dp_2+\ldots+f'(p_n)\,dp_n,$$
$$0 = dp_1 + dp_2 + \ldots + dp_n.$$

Multiply second equation by $f'(p_1)$, and subtract; then, as the variations of the quantities are independent, it follows

$$f'(p_1) = f'(p_2) = \dots = f'(p_n).$$

Since $f'(p)$ is always positive, the above equations cannot hold good unless $p_1 = p_2 = \dots = p_n$ is therefore $= \mathrm{P}/n$; and from the condition that d^2h/dp^2 is negative, it follows that this corresponds to a maximum value.

If the variation is uniform, the sum of happiness will be constant however the property be divided; for in this case an equation of the form $h = cp$ obtains, and $u = c(p_1 + p_2 + \dots + p_n) = c\mathrm{P}$ is constant.

If d^2h/dp^2 is $+$, the solution $p_1 = p_2 = \dots = p_n = \mathrm{P}/n$ corresponds to a minimum.

[This takes no account of the different degrees of happiness afforded to different persons by equal shares of property.]

5644. (A. Martin, LL.D.)—A rectangular hole is cut through the centre of a sphere of radius r; find the average volume removed.

Solution by H. J. Woodall, A.R.C.S.; Professor Krishmachandra De; *and others.*

Let sides of normal section of hole be (a, b); and suppose that the axis of the hole passes through the centre of the sphere, then volume removed is

$$8\int_0^a\int_0^b (r^2 - x^2 - y^2)^{\frac{1}{2}}\, dy\, dx,$$

and the average volume removed is

$$= 8\int_0^r\int_0^{(r^2-u^2)}\ \int_0^u\int_0^v (r^2 - u^2 - v^2)^{\frac{1}{2}}\, dy\, dx,\ dv\,.\,du \Big/ \int_0^r\int_0^{(r^2-u^2)^{\frac{1}{2}}} du\, dv.$$

9167. (Professor B. Hanumanta Rau, B.A.)—Given the coordinates of the centres of four spheres, and their radii, find the coordinates of their radical centre.

Solution by H. J. Woodall, A.R.C.S.; Prof. Radhakrishnan; *and others.*

If (x, y, z) be radical centre, k = common length of tangent.

If (x_1, y_1, z_1) be centre, r_1 the radius of first sphere, and put $x_1^2 + y_1^2 + z_1^2 = s_1^2$, and so with the other spheres, also $x^2 + y^2 + z^2 = s^2$;

$$\therefore\ k^2 + r_1^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2;$$

$$\therefore\ 2(xx_1 + yy_1 + zz_1) = s^2 + s_1^2 - k^2 - r_1^2;$$

and similarly for the other spheres. Whence, by subtraction, we obtain

$$2\left\{x(x_1 - x_2) + y(y_1 - y_2) + z(z_1 - z_2)\right\} = s_1^2 - s_2^2 + r_2^2 - r_1^2 = k_2^2,$$
$$2\left\{x(x_1 - x_3) + y(y_1 - y^3) + z(z_1 - z_3)\right\} = s_1^2 - s_3^2 + r_3^2 - r_1^2 = k_3^2,$$
$$2\left\{x(x_1 - x_4) + y(y_1 - y_4) + z(z_1 - z_4)\right\} = s_1^2 - s_4^2 + r_4^2 - r_1^2 = k_4^2.$$

Solving, we obtain

$$\frac{x}{\begin{vmatrix} k_2^2, & y_1-y_2, & z_1-z_2 \\ k_3^2, & y_1-y_3, & z_1-z_3 \\ k_4^2, & y_1-y_4, & z_1-z_4 \end{vmatrix}} = \frac{y}{\begin{vmatrix} x_1-x_2, & k_2^2, & z_1-z_2 \\ x_1-x_3, & k_3^2, & z_1-z_3 \\ x_1-x_4, & k_4^2, & z_1-z_4 \end{vmatrix}}$$

$$= \frac{z}{\begin{vmatrix} x_1-x_2, & y_1-y_2, & k_2^2 \\ x_1-x_3, & y_1-y_3, & k_3^2 \\ x_1-x_4, & y_1-y_4, & k_4^2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} x_1-x_2, & y_1-y_2, & z_1-z_2 \\ x_1-x_3, & y_1-y_3, & z_1-z_3 \\ x_1-x_4, & y_1-y_4, & z_1-z_4 \end{vmatrix}}.$$

8802. (Rev. T. C. Simmons, M.A.)—In regard to the statement on p. 113 of Vol. xlv. — that "if p denote the chance of A. travelling with one or other of two ladies, whose chances of travelling with him are equal, $\frac{1}{2}p$ will denote his chance of travelling with one in particular"—show that (1) this is true only under certain restrictions; (2) these restrictions do not hold in the case of Quest. 8495; and (3) consequently, it is erroneous to quote the principle as in any way applicable to the solution of that Question.

Solution by the Proposer.

Quest. 8495 is concerned with four passengers A., B., C., D., strangers to one another, journeying in the same railway train. Each of the two ladies C. and D. is as likely to be in any one compartment as in any other; while the two gentlemen A. and B. have a certain equal bias, so that the chance of either being found in a given compartment varies according to the class of the compartment. The phrase "travelling together" being intended to mean "travelling in the same compartment," Quest. 8802 is now easily disposed of.

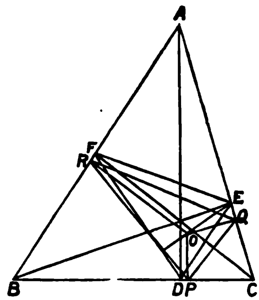
(1) If p denote the chance of the happening of one or the other of two equally likely events, then, according to first principles, the chance of a particular one happening will only be $\frac{1}{2}p$ in the case when the two events are mutually exclusive. For instance, if the train here were to contain an equal number of first and third class compartments, and C. always travelled first class, D. always third class, and A. with equal probability first or third, we should have a case in point, and the $\frac{1}{2}p$ formula would be correct. (2) But this condition cannot hold in Quest. 8495, as the probabilities for C. travelling first, second, or third class are the same as for D. That is to say, the two events considered cannot by any possibility be mutually exclusive. (3) Consequently, Mr. Biddle is entirely in error in applying the principle to the solution of the said Question.

I had been hoping that, in the course of nine years, some other correspondent would have discussed the conclusion arrived at on p. 113 of Vol. xlv. As nobody has done so, perhaps I may be allowed to repeat it; for to me it is both curious and interesting. It is this:—If p denote the chance that A. travels with either lady, and q the corresponding chance for B., then pq will denote the chance, *not that A. and B. both travel with the same lady, but the chance that A. travels with one and B. with the other.*

9305. (Professor B. HANUMANTA RAU, M.A.)—If O be the orthocentre of the pedal triangle of ABC, and OP, OQ, OR the perpendiculars on BC, CA, AB, prove that $\Delta PQR/\Delta ABC$

$$= \tfrac{1}{4}\left(\cos 2B\cos 2C + \cos 2C\cos 2A + \cos 2A\cos 2B - 2\cos 2A\cos 2B\cos 2C\right).$$

Solution by Professors ZERR, MUKHOPADHYAY, *and others.*



$$FE = a\cos A,\quad FD = b\cos B,\quad DE = c\cos C,$$

$$\angle FDE = \pi - 2A,\quad \angle DFE = \pi - 2C,\quad \angle FED = \pi - 2B;$$

$$\therefore\quad EO = \frac{DE\cos 2B}{\sin 2C} = \frac{c\cos C\cos 2B}{\sin 2C} = \frac{c\cos 2B}{2\sin C},$$

$$QO = \frac{EO\cos(A-C)}{2\sin C} = \frac{c\cos 2B\cos(A-C)}{2\sin C}.$$

Similarly $PO = \dfrac{b\cos 2A\cos(C-B)}{2\sin B}$, $RO = \dfrac{a\cos 2C\cos(B-A)}{2\sin A}$,

since $\angle POQ = A+B$, $\angle POR = A+C$, $\angle QOR = BC$,

$$\Delta PQR = \tfrac{1}{8}\left\{\frac{ab\cos 2C\cos 2A\cos(C-B)\cos(B-A)\sin(A+C)}{\sin A\sin B}\right.$$
$$+\frac{ac\cos 2C\cos 2B\cos(B-A)\cos(A-C)\sin(C+B)}{\sin A\sin C}$$
$$\left.+\frac{bc\cos 2B\cos 2A\cos(A-C)\cos(C-B)\sin(A+B)}{\sin B\sin C}\right\};$$

$$\therefore\quad \Delta PQR/\Delta ABC = \tfrac{1}{4}\{\cos 2A\cos 2C\cos(C-B)\cos(B-A)/\sin A\sin C$$
$$+\cos 2C\cos 2B\cos(B-A)\cos(A-C)/\sin C\sin B$$
$$+\cos 2B\cos 2A\cos(A-C)\cos(C-B)/\sin B\sin A.\}$$

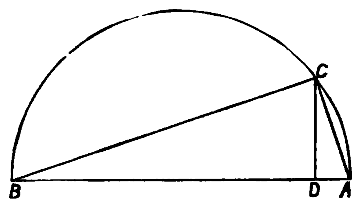
But $\cos(B-A)\cos(C-B)/\sin A\sin C = 1 - \cos B\cos 2B/\sin A\sin C$;

$$\therefore\quad \Delta PQR/\Delta ABC = \tfrac{1}{4}\left\{\cos 2A\cos 2C + \cos 2C\cos 2B + \cos 2B\cos 2A\right.$$
$$\left.-\cos 2A\cos 2B\cos 2C\left(\frac{\cos B}{\sin A\sin C} + \frac{\cos C}{\sin A\sin B} + \frac{\cos A}{\sin C\sin B}\right)\right\}$$
$$= \tfrac{1}{4}\left\{\cos 2A\cos 2C + \cos 2C\cos 2B + \cos 2B\cos 2A\right.$$
$$\left.-\left(\frac{\sin 2A + \sin 2B + \sin 2C}{2\sin A\sin B\sin C}\right)\cos 2A\cos 2B\cos 2C\right\}$$
$$= \tfrac{1}{4}\{\cos 2A\cos 2C + \cos 2C\cos 2B + \cos 2B\cos 2A$$
$$-2\cos 2A\cos 2B\cos 2C\}.$$

12897. (I. Arnold.)—Describe, geometrically, the arc the sum of whose tangent and cotangent is equal to n times the diameter.

Solution by H. W. Curjel, M.A.; V. S. Bouton, B.Sc.; *and others.*

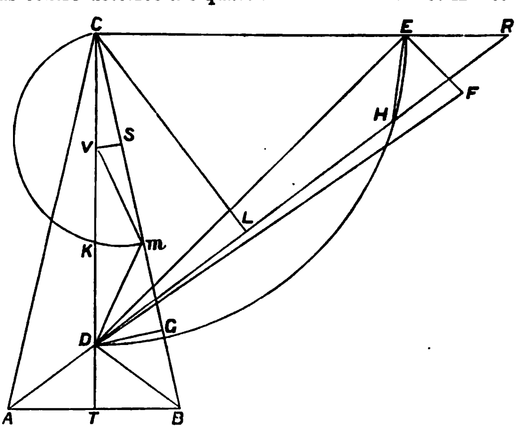
Make AB n times the given diameter, and on it describe a semicircle, and draw a parallel to AB at a distance from it equal to half the given diameter, cutting the semicircle in C. Draw CD perpendicular to AB. Then the angle ACD or DCB is clearly the required angle.



11315. (I. Arnold.)—Given the line (a) drawn to the in-centre from the vertex of an isosceles triangle, each of whose base angles is treble the vertical angle, find (1) the line bisecting the angles at the base; (2) the sides; (3) the bisecting line, and the sides when $a = 32$; and (4) show how any isosceles triangle can be constructed by elementary geometry when the lines drawn from the vertices to the in-centre are given.

Solution by I. Arnold, Professor Krishmachandra De, *and others.*

Let ABC be the isosceles triangle, D the centre of inscribed circle, and CD = a. Draw CE perpendicular to CD and equal to a. Join DE. From C as centre describe the quadrant DHE. Produce AD to meet the



quadrant in H, and CE produced in R. From E draw EF perpendicular to DE and equal to $\frac{1}{2}$AD or $\frac{1}{2}$BD. Join DF.

Draw DG perpendicular to BC, and make $Gm = BG$. Join Dm, and on mC describe the semicircle mKC, and let S be the centre of the semicircle. Erect SV perpendicular to mC. Join Vm.

Now, in the triangle BDC, the angle BCD is $\frac{1}{3}$ the angle DBC. Cm is the difference of the segments CG, BG, and $BD = Dm = mV = VC$. Also $AD = HR$, and $DH = DF - EF$, and $CR = CA = CB$, and $SG = \frac{1}{2}BC$.

Now, if $CD = a$ and BD or $AD = 2x$, then, by simple processes too long to print, we obtain

$$32x^5 + 48ax^4 - 40a^2x^3 - 8a^3x^2 + 8a^4x - a^5 = 0 \text{ .... (1)},$$

or, for $a = 32$,

$$x^5 + 48x^4 - 1280x^3 - 8192x^2 + 262144x - 1048576 = 0 \text{ ...... (2)}.$$

Now the root of equation (2) gives $x = 5{\cdot}71036388$, and, consequently, AD, or BD, or $HR = 11{\cdot}4207276$; and, by substituting this value of x in equation (1), we have

$$BC = 40{\cdot}12677, \quad BC - Cm = Bm = AB;$$

and, by substituting the value of x in this equation, we have

$$AB = 17{\cdot}858.$$

To construct any isosceles triangle geometrically when the bisecting lines CD, AD, BD are given, draw CD, the line bisecting the vertical angle. Draw CE perpendicular to CD and equal to it. Join DE, and describe the quadrant. Erect EF perpendicular to DE and equal to $\frac{1}{4}$AD, the line bisecting the angles at the base. Join DF, and inflect $DH = DF - EF$. Produce DH to meet CE produced in R, and HD till AD becomes equal to HR. From A draw AB parallel to CE, and produce CD to cut it in T. Make $TB = TA$, and join CA, CB. ABC is the isosceles triangle required.

The proof is evident, as it is only requisite to show that HR is equal to 2EF, for then is $LR = LA$, CL being perpendicular to DH, and the angle $CAR = CRA = DAB$; consequently CR is equal to $CA = CB$, and ACB is the isosceles triangle required.

Because the angle $DHE = DER = 135°$, and the angle HDE is common to the two triangles DHE, DER, we have

$$DH \cdot DR = DE^2 = DF^2 - EF^2 \text{ and } DH = DF - EF;$$

we have $DR = DF + EF$, but $DR - DH = 2EF = HR = AD$; $\therefore$ &c.

12641. (Professor Sanjána, M.A.)—Prove that (1) in the solution of Question 2916, the sides of the triangle ABC are as 2 : 5 : 5; (2) a triangle whose sides are as 5 : 10 : 13 (or as 37 : 50 : 85, &c.) has its centre of gravity on the circumference of the in-circle; (3) therefore Mr. Brierley's statement that the triangle of Question 2916 (Vol. lxii., pp. 113, 114) is necessarily isosceles is wrong, and his construction applicable merely to a particular case. Also find the general condition that the problem may be possible. [Professor Sanjána has found the condition analytically; but he does not know if the general problem admits of solution geometrically.]

Solution by C. E. Hillyer, Professor Radhakrishnan, *and others.*

The problem of Question 2916 always admits of a general solution.

For, let AZ, AY be two fixed tangents to a given circle whose centre is I, and BC any third tangent, and G the centre of gravity of ABC.

Let AI meet the circle in P, Q.

If $AY = AZ = k$, we have $a = b+c-2k$;

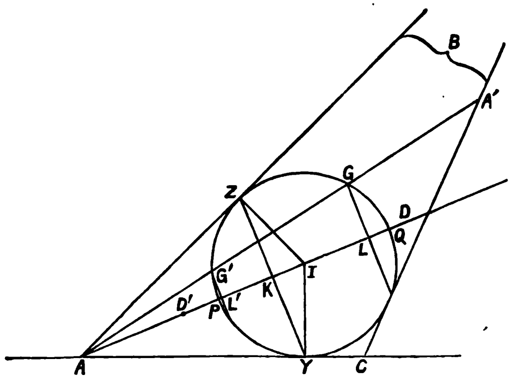
$$\therefore\quad b^2+c^2-2bc\cos A = b^2+c^2+4k^2+2bc-4bk-4ck,$$

$$bc\cos^2\tfrac{1}{2}A-k(b+c)+k^2 = 0 \quad\text{......................}\quad (1).$$

Now, if x, y be the coordinates of G referred to AB, AC,

$$b = 3x,\ c = 3y;\qquad \therefore\quad 9xy\cos^2\tfrac{1}{2}A-3k(x+y)+k^2 = 0$$

is the equation to the locus of G, which is therefore a hyperbola.



Also, it follows from symmetry that the vertices α, α' of the hyperbola are in AI, and $A\alpha = \tfrac{2}{3}AQ$, $A\alpha' = \tfrac{2}{3}AP$.

In order that XYZ may be the inscribed circle, G must be on the further branch, and, since $A\alpha < AQ$, this always cuts the circle, giving two positions of G corresponding to two identical triangles satisfying the conditions.

Further, there will be four positions of G and two solutions if

$$AP \not< \tfrac{2}{3}AQ,\ \textit{i.e.}, \text{ if } \frac{r}{\sin\frac{1}{2}A}-r \not< \tfrac{2}{3}\left(\frac{r}{\sin\frac{1}{2}A}+r\right) \text{ or } \sin\tfrac{1}{2}A \not> \tfrac{1}{5}.$$

Again, if α, β, γ be the perpendiculars from G on YZ, AY, AZ, we have $b = 3\gamma/\sin A$, $c = 3\beta/\sin A$, and if G be on the circle, $\beta\gamma = \alpha^2$.

Substituting in (1) and putting $k = r\cot\tfrac{1}{2}A$,

$$9\beta\gamma-6r(\beta+\gamma)+4r^2\cos^2\tfrac{1}{2}A = 0 \quad\text{......................}\quad (2).$$

Also $$(\beta+\gamma)k-2\alpha r\cos\tfrac{1}{2}A = 2\Delta AYZ = k^2\sin A;$$

$$\therefore\quad \beta+\gamma = r\cot\tfrac{1}{2}A\sin A+2\alpha\sin\tfrac{1}{2}A,$$

whence we obtain by substituting in (2) $9\alpha^2-12r\alpha\sin\tfrac{1}{2}A-8r^2\cos^2\tfrac{1}{2}A = 0$

and $$\alpha = 2r/\left\{\sin\tfrac{1}{2}A \pm (1+\cos^2\tfrac{1}{2}A)^{\frac{1}{2}}\right\} \quad\text{..................}\quad (3).$$

Hence the following construction for G:—Let AI meet YZ in K, in AQ produced take D so that DP . DQ = KY^2, take KL = $\frac{2}{3}$KD, and LG perpendicular to AL to meet the circle in G.

This will always give a real point G, for KL < KQ if $\frac{2}{3}\text{KD} < \text{KI} + r$,

if $2\text{KI} + 2\text{ID} < 3\text{KI} + 3r$, if $\text{ID}^2 < \frac{1}{4}(\text{KI} + 3r)^2$, if $\text{KY}^2 + r^2 < \frac{1}{4}(\text{KI} + 3r)^2$,

if $\quad \text{KY}^2 < \frac{1}{4}(\text{KI} + 5r)(\text{KI} + r)$, if $\text{KP} . \text{KQ} < \frac{1}{4}(\text{KI} + 5r)\,\text{KQ}$,

if $$4\text{KP} < r - \text{KP} + 5r,$$

i.e., if $\quad 5\text{KP} < 6r$, which is always the case, $\quad \because \quad \text{KP} < r$;

and a second point G′ can be found by taking

$$\text{ID}' = \text{ID} \text{ and } \text{KL}' = \tfrac{2}{3}\text{KD}', \text{ provided that } \text{KL}' < \text{KP};$$

i.e., $\quad \frac{2}{3}\text{KD}' < \text{KP}, \quad 2\text{ID}' - 2\text{IK} < 3r - 3\text{IK}, \quad \text{ID}'^2 < \frac{1}{4}(3r - \text{IK})^2,$

$$r^2 + \text{KY}^2 < \tfrac{1}{4}(3r - \text{IK})^2, \quad \text{KY}^2 < \tfrac{1}{4}(5r - \text{IK})(r - \text{IK}),$$

$$\text{PK} . \text{KQ} < \tfrac{1}{4}(5r - \text{IK})\,\text{PK}, \quad 4\text{IK} + 4r < 5r - \text{IK};$$

i.e., $$5\text{IK} < r, \text{ or } \sin\tfrac{1}{2}\text{A} < \tfrac{1}{5}.$$

The relation between the sides of a triangle in order that its centre of gravity may be on the inscribed circle is found from (3), for

$$\beta\gamma = \tfrac{1}{9}(bc\sin^2\text{A}) \text{ and } a^2 = \tfrac{1}{9}(8r^2)\left\{1 \pm (1 - \cos^4\tfrac{1}{2}\text{A})^{\frac{1}{2}}\right\};$$

$$\therefore \quad bc\sin^2\text{A} = 8r^2\left\{1 \pm (1 - \cos^4\tfrac{1}{2}\text{A})^{\frac{1}{2}}\right\},$$

and $$r = (bc\sin\text{A})/2s, \quad \cos^2\tfrac{1}{2}\text{A} = s(s-a)/bc;$$

whence the relation may be written

$$5s^2 - 8a . s + 4a^2 - 4bc = 0, \text{ or } 5(a^2 + b^2 + c^2) = 6(bc + ca + ab).$$

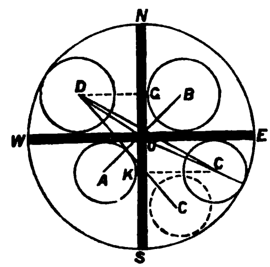
It will be found by trial that

$$a = 5, \quad b = 10, \quad c = 13, \quad \text{or} \quad a = 37, \quad b = 50, \quad c = 85 \text{ satisfy.}$$

11794. (Professor SHIELDS.)—A queen with four children, A., B., C., and D., owned a *round* island of land with two cross streets, each 4 rods wide, running north and south, and east and west, dividing the island into four equal quadrants. She gave A. a round tract of land in the S.W. quadrant, tangent to both streets, and enclosing two acres of land in the corner between the circumference of the land and junction of the two streets; and gave B. a similar *round* tract of land in the N.E. quadrant, tangent to the two streets, thus enclosing 3 acres of land in the corner outside of and between the round tract and two streets. She then gave C. a *round* tract of land in the S.E. quadrant, tangent to *one* street and circumference of the island, in which was the largest *square* field possible, enclosing 2 acres of land in each of the four segments outside of the *square* field; and she gave D. a similar round tract in the N.W. quadrant, tangent to *one* street and circumference of the island, in which was the largest *square* field possible, enclosing 3 acres of land in each of the four segments outside of the *square* field. The centres of opposite round tracts are connected by two diagonal lines, AB and CD. And, knowing that the sum of the circumferences of the four round tracts is equal to the circumference of the island, find (1) the area of the island, (2) the number of acres each

child received, and (3) the difference in the lengths of the two diagonals AB and CD.

Solution by Professors Zerr, Nilkantha Sarkar, *and others.*



Let O be the centre of the island, K, G points in the middle of the street NS, so that DG, CK are perpendicular to the street, R the radius of the island, r_1, r_2, r_3, r_4 the radii of the tracts given to A., B., C., D. respectively; then

$r_1^2 - \frac{1}{4}\pi r_1^2 = 2$ acres $= 320$ sq. rods;
$\therefore\ r_1 = 38{\cdot}61536 +$ rods.
$r_2^2 - \frac{1}{4}\pi r_2^2 = 3$ acres $= 480$ sq. rods;
$\therefore\ r_2 = 47{\cdot}29397 +$ rods.
$\pi r_3^2 - 2r_3^2 = 8$ acres $= 1280$ sq. rods;
$\therefore\ r_3 = 33{\cdot}48482 +$ rods.
$\pi r_4^2 - 2r_4^2 = 12$ acres $= 1920$ sq. rods; $\quad \therefore\ r_4 = 41{\cdot}01036 +$ rods.

$R = r_1 + r + r_3 + r_4 = 160{\cdot}40451$ rods. Hence, in acres,

$$\pi R^2 = 505{\cdot}200830074, \quad \pi r_1^2 = 29{\cdot}278657968, \quad \pi r_2^2 = 43{\cdot}917986952,$$
$$\pi r_3^2 = 22{\cdot}015416959, \quad \pi r_4^2 = 33{\cdot}023125438;$$
$$AB = (r_1 + r_2 + 4)\sqrt{2} = 127{\cdot}15067 + \text{rods},$$
$$DG = \{(R - r_4)^2 - (r_4 + 2)^2\}^{\frac12} = 111{\cdot}37806,$$
$$CK = \{(R - r_3)^2 - (r_3 + 2)^2\}^{\frac12} = 121{\cdot}85826,$$

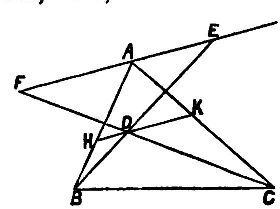
$CD = \{(DG + CK)^2 + (r_3 + r_4 + 4)^2\}^{\frac12} = 246{\cdot}09078 +$ rods when C and D are tangent to the same street;

$CD = \{(DG + r_3 + 2)^2 + (CK + r_4 + 2)^2\} = 220{\cdot}79485 +$ rods when C and D are tangent to different streets;

$\therefore\ CD - AB = 118{\cdot}94011 +$ rods or $93{\cdot}64418 +$ rods.

12998. (H. D. Drury, M.A.)—To draw across a triangle a line in a given direction, such that the portion of the line intercepted by the sides may bear to the sum of the lower segments of the sides a given ratio.

Solution by I. Arnold; H. W. Curjel, M.A.; *and others.*



Draw FAE in the given direction through A. Cut off FA, FE, so that

FA : AC = AE : AB = given ratio.

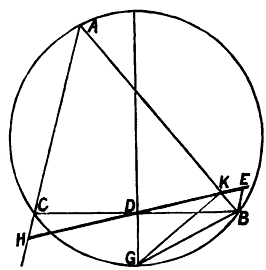
Through D, the intersection of BE, CF, draw HK cutting AB, AC in H and K.

Then HK is clearly the required straight line.

12979. (Professor SÇHWATT.)—(1) If from the middle point M of the side BC of the triangle ABC a parallel to the bisector AF of the external angle to ABC is drawn to meet AB in K, the point K divides the side AB in KA = $\frac{1}{2}$(AB + BC) and KB = $\frac{1}{2}$(AB − AC). (2) If K is joined to the extremity D of the diameter perpendicular to BC, then is KD perpendicular to AB.

Solution by Professor NATH COONDOO; H. W. CURJEL, M.A.; *and others.*

Draw BE parallel to AC, cutting DK in E. Let DK cut AC in H, and G be the end of the diameter of the circumcircle which is perpendicular to BC.



Then $\quad$ CH = BE = KB,

and $\quad$ AH = AK;

$\therefore$ $\quad$ AK = $\frac{1}{2}$(AB + AC),

KB = $\frac{1}{2}$(AB − AC).

(2) Also DKGB are concyclic, for

$\angle$AKD = complement of $\frac{1}{2}$A = $\angle$BGD;

$\therefore$ $\quad$ $\angle$GKB = $\angle$BDG = a right angle.

[As a specimen of the use of vectors, the following solution has been sent by Rev. D. THOMAS:—

Let α, β, γ be the vectors of A, B, C respectively drawn from O, the circumcentre, and a, b, c as usual the sides of the triangle ABC.

The direction of MK is given by $(\alpha-\beta)/c+(\gamma-\alpha)/b$, and OK is

$$\tfrac{1}{2}(\beta+\gamma)+k\left[(\alpha-\beta)/c+(\gamma-\alpha)/b\right], \text{ if } \tfrac{1}{2}+k/b=0,\ k=-\tfrac{1}{2}b;$$

or
$$\text{OK}=\tfrac{1}{2}\beta-\tfrac{1}{2}b\left(\frac{\alpha-\beta}{c}-\frac{\alpha}{b}\right)=\frac{(c+b)\,\beta+(c-b)\,\alpha}{2c},$$

which shows that AB is divided in the required manner.

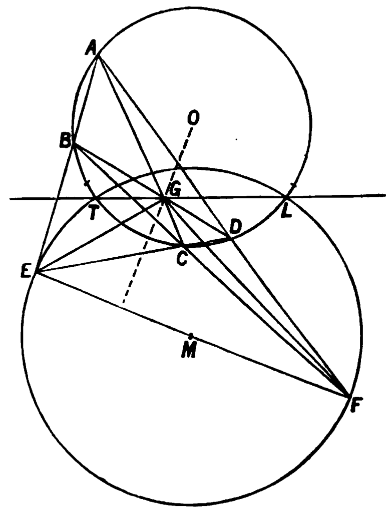
$$(2)\qquad \text{KD}=\text{KB}+\text{BD};$$

$$\begin{aligned}\therefore\quad \text{S}.\text{AB}.\text{KD} &= \text{S}.\text{AB}.\text{KB}+\text{S}.\text{AB}.\text{BD}\\ &= -\tfrac{1}{2}(c^2-bc)+2\text{R}c.\cos(\text{B}+\tfrac{1}{2}\text{C})\sin\tfrac{1}{2}\text{A}\\ &= -\tfrac{1}{2}(c^2-bc)+\text{R}c\sin(\text{A}+\text{B})-\text{R}c\sin\text{B}\\ &= -\tfrac{1}{2}(c^2-bc)+\tfrac{1}{2}c^2-\tfrac{1}{2}bc=0,\end{aligned}$$

which shows that AB and KD are at right angles.]

12982. (M. VERRIÈRE.)—On considère une circonférence O et un point extérieur M, et tous les quadrilatères inscrits dans la circonférence donnée et tel que le point M soit le milieu de leur troisième diagonale EF. G étant le point de concours des deux autres diagonales, on demande (1) de trouver le lieu du centre de gravité du triangle EFG, et (2) de déterminer l'enveloppe des droites GE et GF.

Solution by Professor A. Droz-Farny; H. W. Curjel, M.A.; *and others.*



Soit TL la polaire de M par rapport au cercle O. G étant le pole de la diagonale EF, cette dernière tournant autour du point M, G décrit la droite TL. Le centre de gravité du triangle EFG, divisant la médiane MG dans le rapport fixe GS : SM = 1 : 2, le lieu de ce point est une parallèle à TL.

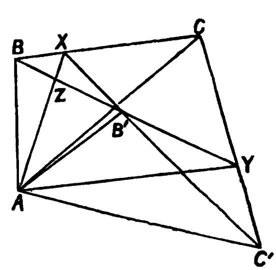
On sait que la circonférence décrite sur EF comme diamètre coupe orthogonalement le cercle O; les points E et F décrivent donc une circonférence fixe de centre M et de rayon

ME = MF = MT = ML.

Les droites GE et GF polaires des points F et E enveloppent donc la réciproque polaire de la circonférence M par rapport à O. C'est une hyperbole ayant en O un de ses foyers; les directions asymptotiques sont perpendiculaires aux rayons OT et OL.

12968. (Professor Genese, M.A.)—A triangle ABC is rotated round A in its plane into any other position AB′C′. If BC, B′C′ meet in X, and BB′, CC′ in Y, then angle XAY = B ∼ C.

Solution by Rev. J. L. Kitchin; H. W. Curjel, M.A.; *and others.*



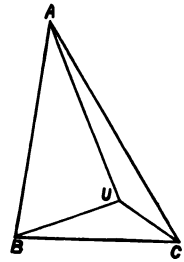
Let BB′, AX cut in Z.

Then B, X, B′, A are concyclic, and also C, C′, A, X and B′, A, Y, C′;

$$\therefore\quad \angle C + \angle XAY = \angle AYZ + \angle ZAY$$
$$= \angle AZB = \pi - \angle B'BA - XAB$$
$$= \pi - XB'A = B' = B;$$
$$\therefore\quad \angle XAY = B - C.$$

12997. (J. GRIFFITHS, M.A.)—If, in a triangle ABC, a point U be taken so that $\angle UBC = \omega = \angle UCA$, and $\angle AUC = \pi - \theta$, prove that $\cot\omega = \cot\theta + \cot B + \cot C$.

Solution by H. W. CURJEL, M.A.; Prof. A. DROZ-FARNY; *and others.*



Here we have $\angle CAU = \theta - \omega$.

Hence

$$\sin^2\omega \sin(B + C + \theta - \omega) = \sin(\theta-\omega)\sin(B-\omega)\sin(C-\omega);$$

$$\therefore \quad \sin(B+C)\cot(\theta-\omega) + \cos(B+C) = (\sin B\cot\omega - \cos B)(\sin C\cot\omega - \cos C);$$

$$\therefore \quad (\cot B + \cot C)\cot(\theta-\omega) + \cot B\cot C - 1 = \cot^2\omega - \cot\omega(\cot B + \cot C) + \cot B\cot C;$$

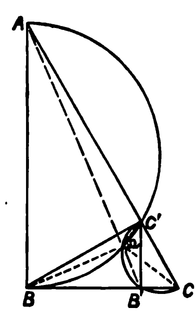
$$\therefore \quad (\cot B + \cot C)\left\{\frac{\cot\omega\cot\theta + 1}{\cot\omega - \cot\theta} + \cot\omega\right\} - \cot^2\omega - 1 = 0;$$

$$\therefore \quad (\cot^2\omega + 1)(\cot B + \cot C - \cot\omega + \cot\theta) = 0;$$

$$\therefore \quad \cot\omega = \cot\theta + \cot B + \cot C.$$

12970. (Professor SANJÁNA.)—In a triangle right-angled at B, BC′ is drawn perpendicular to AC, and C′B′ to BC. Prove that the triangles ABC and BB′C′ have the same positive BROCARD point, and that this point lies on AB′.

Solution by Professors A. DROZ-FARNY, GOPALUCHANAR, *and others.*



Les circonférences circonscrites aux triangles ABC′ et CC′B′ se coupent en Ω. On a

$$\sphericalangle A\Omega C' = ABC' = C \quad \text{et} \quad C'\Omega B' = 180° - C.$$

Les points A, Ω, B′ sont donc en ligne droite.

La circonférence ABC′ étant tangente en B à BC, il en résulte

$$\sphericalangle \Omega BC = BA\Omega = \Omega B'C,$$

et dans le quadrilatère inscriptible $B'\Omega C'C$ on trouve $\sphericalangle BC'\Omega = C'C\Omega = C'B'\Omega$.

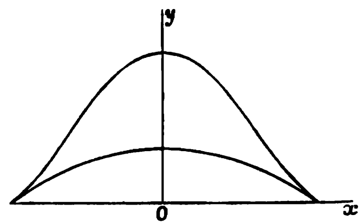
Par conséquent,

$$\Omega AB = \Omega BC = \Omega CA,$$
$$\Omega BB' = \Omega B'C' = \Omega C'B;$$

d'où la thèse.

12978. (Professor MATZ.)—The closed portion of the curve known as "the Cocked Hat," $x^4+x^2y^2+4ax^2y-2a^2x^2+3a^2y^2-4a^3y+a^4=0$, revolves round the axis of y. Find (1) the *campanulate* volume generated; if the same portion of the curve revolve round the axis of x, find the *fusiform* volume generated. Also determine the area of this closed portion of the curve.

Solution by H. W. CURJEL, M.A.; Prof. RADHAKRISHNAN; *and others.*



The equation may be written

$$y=\frac{(a^2-x^2)\left\{2a\pm(a^2-x^2)^{\frac12}\right\}}{x^2+3a^2};$$

therefore area of curve

$$=4\int_0^a\frac{(a^2-x^2)^{\frac32}}{x^2+3a^2}\,dx$$

$$=4a^2\int_0^{\frac12\pi}\frac{\cos^4\theta\,d\theta}{4-\cos^2\theta}$$

$$=4a^2\int_0^{\frac12\pi}\left(-\cos^2\theta-4+\frac{16}{4-\cos^2\theta}\right)d\theta$$

$$=4a^2\left\{-\tfrac14\pi-2\pi+4\pi/\sqrt3\right\}=\tfrac13(\pi a^2)\left\{16\sqrt3-27\right\}.$$

Also campanulate volume

$$=4\pi\int_0^a\frac{(a^2-x^2)^{\frac32}}{x^2+3a^2}\,x\,dx$$

$$=4\pi a^3\int_0^{\frac12\pi}\left(-\cos^2\theta-4+\frac{4}{2-\cos\theta}+\frac{4}{2+\cos\theta}\right)\sin\theta\,d\theta$$

$$=\frac{4\pi a^3}{3}(12\log3-13).$$

And fusiform volume

$$=2\pi\int_0^a\frac{(a^2-x^2)^2}{(x^2+3a^2)^2}\,8a\,(a^2-x^2)^{\frac12}\,dx$$

$$=16\pi a^3\int_0^{\frac12\pi}\frac{\cos^6\theta}{(4-\cos^2\theta)^2}\,d\theta$$

$$=16\pi a^3\int_0^{\frac12\pi}\left\{\cos^2\theta+8-\frac{80+128\tan^2\theta}{(4\tan^2\theta+3)^2}\sec^2\theta\right\}d\theta$$

$$=68\pi^2a^3-128\pi a^3\int_0^\infty\frac{5+2z^2}{(z^2+3)^2}\,dz$$

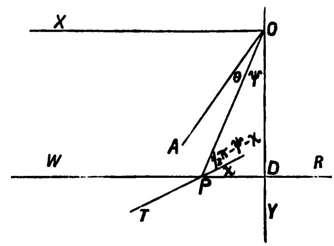
$$=68\pi^2a^3-128\pi a^3\int_0^\infty\frac{2dz}{z^2+3}+128\pi a^3\int_0^\infty\frac{dz}{(z^2+3)^2}$$

$$=68\pi^2a^3-\frac{128\pi^2a^3}{\sqrt3}+\frac{32}{3\sqrt3}\pi^2a^3$$

$$=\tfrac19(\pi^2a^3)(612-352\sqrt3).$$

12990. (W. C. STANHAM.)—The floats of a paddle-wheel of a steamer enter the water without splashing, the angular velocity of the wheel (ω) and the velocity of the steamer (v) being constant. Find the polar equation of the curve given by a section of the surface of the float perpendicular to the axis of the wheel.

Solution by H. ORFEUR, Professor RADHAKRISHNAN, *and others.*



So that there may be no splashing, the required section must, at the point on the surface of the water, touch the path described by that point through the motion of the steamer and the rotation of the wheel. Let O be the centre of the wheel, WR the surface of the water, OD the perpendicular from O on to WR. Let $OD = a$. Taking O for origin, the direction in which the steamer moves for OX, and OD for OY, we get for the locus of S a point on the wheel at a distance $a\sec\psi$ from O,

$$x = a\sec\psi\cos\phi + (v/\omega)\,\phi, \quad y = a\sec\psi\sin\phi; \quad \text{where} \quad \phi = \angle \text{SOX}.$$

For this curve, $$\frac{dy}{dx} = \frac{\omega a\sec\psi\cos\phi}{v - \omega a\sec\psi\sin\phi},$$

Suppose P is the position of S when it is in WR; then

$$\angle \text{POD} = \psi \quad \text{and} \quad \phi = \tfrac{1}{2}\pi - \psi$$

Hence, at P, $$\frac{dy}{dx} = \frac{\omega a\tan\psi}{v - \omega a} = \tan\chi, \text{ suppose.}$$

Through P draw PT, so that $\angle \text{WPT} = \chi$.

Then PT touches the required section;

$$\therefore \quad r\frac{d\theta}{dr} = \tan\left\{\tfrac{1}{2}\pi - (\psi + \chi)\right\} = \frac{v - \omega a\sec^2\psi}{v\tan\psi};$$

r and θ being the polar coordinates of P, O being the pole.

Now $$r = a\sec\psi;$$

$$\therefore \quad r\frac{d\theta}{dr} = \cot\psi\frac{d\theta}{d\psi}; \qquad \therefore \quad \frac{d\theta}{d\psi} = \frac{1}{v}(v - \omega a\sec^2\psi);$$

$$\therefore \quad \theta = \psi - (\omega a/v)\tan\psi + \beta, \text{ where } \beta \text{ is some constant.}$$

If A is the inside extremity of the section, $OA = a$. Take OA as the initial vector. Then $\angle \text{AOP} = \theta$. For point A, $\psi = 0$ and $\theta = 0$;

$$\therefore \quad \beta = 0.$$

Hence required equation is obtained by eliminating ψ between

$$r = a\sec\psi, \quad \theta = \psi - (\omega a/v)\tan\psi.$$

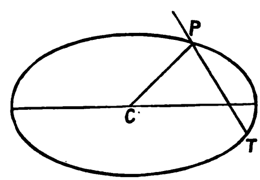
The polar equation is $\theta = \sec^{-1}(r/a) - (\omega/v)(r^2 - a^2)^{\frac{1}{2}}$.

12701. (Professor Sanjána, M.A.)—Chords are drawn from the end of central radii of the ellipse $x^2/a^2+y^2/b^2=1$ at right angles to the radii; show that (1) the locus of their mid-points is

$$(a^2y^2+b^2x^2)(a^6y^2+b^6x^2)^{\frac{1}{2}} = a^5by^2+b^5ax^2;$$

and (2) how the problem may be solved when the chords make *any* fixed angle with the radii.

Solution by Prof. Radhakrishnan; Rev. J. L. Kitchin, M.A.; *and others.*



If the equation to the chord be

$$lx+my=1,$$

then the equation to the perpendicular on it from the centre is

$$mx-ly=0,$$

and the point of intersection is

$$\left(\frac{l}{l^2+m}, \frac{m}{l^2+m^2}\right).$$

As this point is on the ellipse by hypothesis, we have

$$\frac{l^2}{a^2}+\frac{m^2}{b^2}=(l^2+m^2)^2 \quad \text{.........................} \quad (1).$$

Now, taking the equations $lx+my=1$ and $x^2/a^2+y^2/b^2=1$, and eliminating y or x, we get the equations

$$\frac{m^2x^2}{a^2}+\frac{(1-lx)^2}{b^2}=m^2, \quad \frac{(1-my)^2}{a^2}+\frac{l^2y^2}{b^2}=l^2.$$

Therefore the coordinates of the middle point of the chord (say x, y) equal to half the sum of the roots are

$$\frac{la^2}{a^2l^2+b^2m^2} \text{ and } \frac{mb^2}{a^2l^2+b^2m^2};$$

$$\therefore \quad \frac{x}{y}=\frac{la^2}{mb^2}; \qquad \therefore \quad l=\frac{mb^2}{a^2}\cdot\frac{x}{y} \quad \text{.....................} \quad (2).$$

$$y=\frac{mb^2}{a^2b^2+b^2m^2}=\frac{mb^2}{(m^2b^4/a^2)\cdot(x^2/y^2)+b^2m^2}=\frac{a^2y^2}{m(b^2x^2+a^2y^2)};$$

$$\therefore \quad m=\frac{a^2y}{b^2x^2+a^2y^2} \quad \text{..........................} \quad (3).$$

Substituting (2) and (3) in (1), we get

$$\frac{b^4}{a^6}\cdot\frac{x^2}{y^2}+\frac{1}{b^2}=\frac{a^4y^2}{(b^2x^2+a^2y^2)^2}\left(\frac{b^4}{a^4}\cdot\frac{x^2}{y^2}+1\right)^2;$$

$$\therefore \quad (a^6y^2+b^6x^2)(b^2x^2+a^2y^2)^2 = a^2b^2(b^4x^2+a^4y^2)^2,$$

i.e., $$(a^6y^2+b^6x^2)^{\frac{1}{2}}(b^2x^2+a^2y^2) = ab^5x^2+ba^5y^2.$$

(2) If the equation to the radius making the fixed angle with the chord $lx+my=1$ be $x+ny=0$, then $(m-ln)/(l+mn)=$ tangent of the fixed angle $=$ a constant k; therefore n is known in terms of l and m.

If x and y be found from the equations $lx+my=1$ and $x+ny=0$,

and substituted in the equation to the ellipse, we get some other relation between l and m, corresponding to (1).

The remaining part of the process being the same as before, and the values of l and m being found in terms of the coordinates of the middle point of the chord, from the equations (2) and (3), we have simply to substitute these values in the relation between l and m, corresponding to equation (1), to find out the locus.

12941. (Professor GRUBER.)—Find the first six integral values of n in $\frac{1}{2}n(n+1) = \square$.

Solution by R. F. MUIRHEAD, M.A.; H. W. CURJEL, M.A.; *and others.*

The solutions are given by the solutions of $x^2-2y^2 = +1$, which give

$$n = 0, 1, 8, 49, 288, 1681, 9800, \&c.$$

12946. (EDITOR.)—Solve the equations

$$x-z = 12, \quad (x+y+z)\,x = 299, \quad (x+y+z)(y+z) = 230.$$

Solution by Rev. S. J. ROWTON, M.A.; Prof. NATH COONDOO; *and others.*

From (2), (3) we get $(x+y+z)^2 = 529$; $\quad\therefore\quad x+y+z = \pm 23$.

Therefore, from (2), $x = \pm 13$, whence $y = 9$ or 15, $z = 1$ or -25.

8645 & 12864. (Rev. T. C. SIMMONS, M.A.)—If G be the centroid of a triangle ABC, and another triangle $A_1B_1C_1$ be formed with sides respectively equal to $\sqrt{3}\,.\,GA$, $\sqrt{3}\,.\,GB$, $\sqrt{3}\,.\,GC$, prove (1) that ABC may be derived from $A_1B_1C_1$ in the same way as the latter was derived from the former, that is to say, the relation between the triangles is a conjugate one; (2) that their areas are equal, as also their Brocard-angles; (3) that their Lemoine-radii (both first and second), cosine-radii, axes of Brocard-ellipse (major and minor), as well as the distances of their circumcentres from their several symmedian-lines, are all to each other in the ratio of their circumradii; (4) that hence, if in the circum-circle of ABC a triangle A′B′C′ be inscribed similar to $A_1B_1C_1$, then ABC, A′B′C′ can be superposed in such wise as to have their circum-circles, first and second Lemoine-circles, cosine-circles, B.-ellipses, and Lemoine-points coincident, and their symmedians collinear each with each; (5) that in this case $\triangle A'B'C' : \triangle ABC$

$$= 27a^2b^2c^2 : (2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2).$$

[The triangles mentioned in (4) have been called *Co-symmedian.*]

Solution by Professors SANJÁNA, RAMASWAMI AIYAR, *and others.*

This question is discussed in MILNE's *Companion to Problem Papers;* see pp. 147–151, and especially the examples on p. 151.

[The question was originally proposed as 8645, in July, 1886, and revised in February, 1886, two years before the publication of MILNE's *Companion.* It must be borne in mind that such things often happen with regard to Questions proposed in our columns.]

12942. (Professor YOUNG.)—Prove that (1) $\frac{1}{6}n(n+1)(2n+1)$ is a whole number for all values of n, and (2) $\frac{1}{24}(n-1)n(n+1)$ is a whole number when n is odd.

Solution by Rev. D. THOMAS, M.A.; R. W. D. CHRISTIE; *and others.*

We know that the product of n consecutive numbers is divisible by n!. Hence, as

$$(1)\quad n(n+1)(2n+1) = (n-1)n(n+1)+n(n+1)(n+2),$$

it is divisible by 3 ! or 6.

$$\begin{aligned}(2)\quad (n-1)n(n+1) &= 2m(2m+1)(2m+2) \quad \text{if } n = 2m+1\\ &= 4\left[(m-1)m(m+1)+m(m+1)(m+2)\right]\\ &= 4\times\text{multiple of } 6 = \text{multiple of } 24.\end{aligned}$$

9224. (F. MORLEY, B.A.)—Find $\int_0^{2-\sqrt{2}} \log\frac{1-x}{1-\frac{1}{2}x}\frac{dx}{x}$.

Solution by the PROPOSER.

Let $(1-x)/(1-bx) = y$, so that x and y are symmetrically related

Then, if $\qquad \mathrm{I}_{x_0}^{x}$ denotes $\int_{x_0}^{x}\log y\, d\log x$,

we have $\qquad \mathrm{I}_{x_0}^{x}+\mathrm{I}_{y_0}^{y} = \log x\log y-\log x_0\log y_0,$

and in particular if a be a root of $(1-x)/(1-bx) = x$.

$$\mathrm{I}_0^{a}+\mathrm{I}_1^{a} = (\log a)^2, \quad \text{since, when } x = 0,\ \text{lim.}\log x\log(1-x) \text{ is } 0.$$

Now $\qquad \mathrm{I}_0^1 = \int_0^1 \log(1-x)\,dx/x-\int_0^1\log(1-bx)\,dx/x.$

The former integral is $-\pi^2/6$; the latter is $\int_0^b\log(1-x)\,dx/x$.

This is known when $b = \frac{1}{2}$ (it is given in BERTRAND's *Calcul Intégrale,* p. 217, and is also easily deduced from Ex. 3, p. 199, of HARKNESS and MORLEY's *Theory of Functions*). Its value then is $\frac{1}{2}(\log 2)^2-\pi^2/12$;

$$\therefore\quad \mathrm{I}_0^1 = -\pi^2/12-\tfrac{1}{2}(\log 2)^2.$$

When $b = \frac{1}{2}$, we have $1-2a+\frac{1}{2}a^2 = 0$ and $a = 2 \pm \surd 2$.

Therefore $$2\,\mathrm{I}_0^{2-\surd 2} - \mathrm{I}_0^1 = \{\log(2-\surd 2)\}^2,$$

and $$\mathrm{I}_0^{2-\surd 2} = \tfrac{1}{2}\{\log(2-\surd 2)\}^2 - \pi^2/24 - \tfrac{1}{4}(\log 2)^2.$$

[The correctness of the result may be verified by means of the approximate value given by Mr. Woodall, on p. 65 of Vol. LXIII.]

12727. (J. J. Barniville, B.A.)—Prove that $\tan 20^\circ + 4\sin 20^\circ = \surd 3$.

Solution by R. Chartres; Rev. S. J. Rowton, M.A., Mus.D.; *and others.*

Since $\dfrac{\sin 40^\circ + (\sin 40^\circ + \sin 20^\circ)}{\cos 20^\circ} = \surd 3$, $\therefore$ $\tan 20^\circ + 4\sin 20^\circ = \surd 3$.

12953. (Rev. S. J. Rowton, M.A., Mus.D.)—A. has five threepenny loaves, B. three, and C. none. They share equally and eat all the loaves. C. then puts down eight pennies, and goes. How ought A. and B. to divide the money?

Solution by J. McCubbin, B.A.; Prof. Krishmachandra De; *and others.*

Each man eats $2\frac{2}{3}$ loaves. Therefore A. gives up $2\frac{1}{3}$ loaves, and B. $\frac{1}{3}$ loaf, the values of which are $7d.$ and $1d.$ respectively.

10370. (W. J. C. Sharp, M.A.)—If α and β are any two numbers, show that the sum of the homogeneous products of $\alpha^r, \alpha^{r-1}\beta, \alpha^{r-2}\beta^2 \ldots \beta^r$, s together, is the same as the sum of the homogeneous products of $\alpha^s, \alpha^{s-1}\beta, \alpha^{s-2}\beta^2 \ldots \beta^s$, r together.

Solution by H. J. Woodall, A.R.C.S.; Prof. Gopalachanar; *and others.*

The function containing the homogeneous products, p_1, p_2, &c., of $a, b, c, \ldots k$ is

$$\{(1-ax)(1-bx)(1-cx)\ldots(1-kx)\}^{-1} = 1 + p_1 x + p_2 x^2 + \ldots.$$

In the given case, we have

$$\{(1-\alpha^r x)(1-\alpha^{r-1}\beta x)(1-\alpha^{r-2}\beta^2 x)\ldots(1-\beta^r x)\}^{-1} = 1 + \mathrm{S}_{r,1}x + \mathrm{S}_{r,2}x^2 + \ldots.$$

We may get the $(r+1)$th series from this in either of the following ways:— first, change x into αx, and divide by $(1-\beta^{r+1}x)$;
second, change x into βx, and divide by $(1-\alpha^{r+1}x)$.

Therefore $\quad (1-\alpha^{r+1}x)(1+S_{r,1}\,\alpha x+S_{r,2}\,\alpha^2x^2+\ldots)$
$$=(1-\beta^{r+1}x)\;(1+S_{r,1}\,\beta x+S_{r,2}\,\beta^2x^2+\ldots).$$

Take coefficient of x^l on both sides;
$$\therefore\quad \alpha^l\,S_{r,l}-\alpha^{r+l}\,S_{r,l-1}=\beta^l\,S_{r,l}-\beta^{r+l}\,S_{r,l-1};$$
$$\therefore\quad S_{r,l}:S_{r,l-1}=(\alpha^{r+l}-\beta^{r+l}):(\alpha^l-\beta^l).$$

Thus we get $\quad S_{r,l}=\Pi_1{}^l\left\{\alpha^{r+k}-\beta^{r+k})/(\alpha^k-\beta^k)\right\},$
$$S_{r,s}=\Pi_1{}^s\left\{(\alpha^{r+k}-\beta^{r+k})/(\alpha^k-\beta^k)\right\}$$
$$=\Pi_1{}^{r+s}(\alpha^k-\beta^k)/\left\{\Pi_1{}^r(\alpha^k-\beta^k)\,\Pi_1{}^s(\alpha^k-\beta^k)\right\}=S_{s,r};$$

whence the theorem.

12956. (J. O'Byrne Croke, M.A.)—Find, by the use of a general theorem of relation, x, y, z from
$$x^2-yz=a,\quad y^2-zx=b,\quad z^2-xy=c.$$

Solution by Rev. S. J. Rowton, M.A.; Prof. A. Droz-Farny; *and others.*

We have
$$\frac{x}{a^2-bc}=\frac{y}{b^2-ca}=\frac{z}{c^2-ab}=k,\text{ say};$$
$$\therefore\quad x^2-yz=a=k^2\left\{(a^2-bc)^2-(b^2-ca)(c^2-ab)\right\},$$

whence $\quad k=(a^3+b^3+c^3-3abc)^{-\frac{1}{2}},$

and the values of x, y, z follow at once.

8721. (By Professor Ignacio Beyens.)—Résoudre le système d'équations: $\quad x^4+a-b=y^4+c-d=z^4-a-c=u^4+b+d=xyzu.$

Solution by H. J. Woodall, A.R.C.S.; Prof. Chakrivarti; *and others.*

Put each member of the equation $=t$; then we have
$$x=(t-a+b)^{\frac{1}{4}},\quad y=(t-c+d)^{\frac{1}{4}},\quad z=(t+a+c)^{\frac{1}{4}},\quad u=(t-b-d)^{\frac{1}{4}},$$
$$xyzu=t=\&c.$$

Take the fourth power of both sides; then we have
$$t^4=(t-a+b)(t-c+d)(t+a+c)(t-b-d).$$

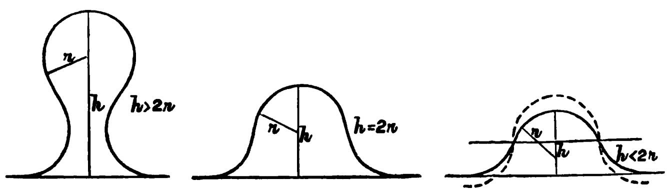
Expanding, we get
$$t^2(ab+cd-ac-bd-a^2-b^2-c^2-d^2)+t\,(a+c-b-d)(a+d)(b+c)$$
$$-(a-b)(c-d)(a+c)(b+d)=0,$$

a quadratic in t; hence there are two values of each variable.
Thus two solutions.

3813. (Professor HUDSON, M.A.)—All vertical sections of a hill from the base to the summit are alike, and consist of two equal arcs of equal circles of which the lower has its convexity downwards and the upper has its convexity upwards, the highest and lowest tangents being horizontal; find whether a person who goes right over it or half round it traverses the greater distance. If the height of the hill be equal to the radius of either circle, find its apparent angular elevation from the base, and the height of equal towers at the base and summit the tops of which are just mutually visible.

Solution by H. W. CURJEL, M.A.; Prof. NATH COONDOO; *and others.*

From the figure it is clear that the path over the hill is >, =, or < the path round it, according as the height of hill is >, =, or < the diameter of circle. If the height of the hill = radius of circle, the angle of elevation of the top of the hill from the base = 30°.

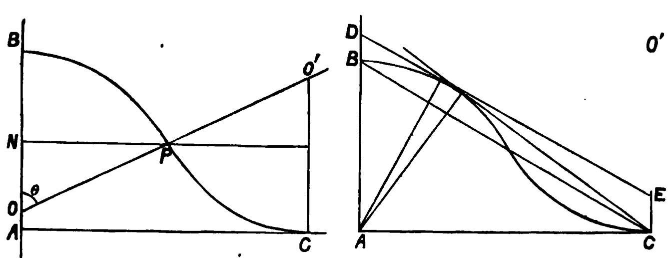


Therefore the line joining the tops of the towers makes an angle 30° with the horizon. Therefore the height of the towers is

$$r(\sec 30^\circ - 1) = \tfrac{1}{3}r(2\sqrt{3}-3).$$

Length of tangent from foot of hill to the hill $= r\sqrt{2}$; hence the apparent elevation $= \cos^{-1}\sqrt{2}/\sqrt{3}$.

[The PROPOSER gives this Solution:—Let BPC be a section of the hill, O, O′ the centres of the circles which touch at P, BA vertical, AC horizontal, BOP $= \theta$, PN perpendicular to OB. Half-round > right-over if $\pi 2r\sin\theta > 4r\theta$, that is, if $\frac{1}{2}\pi > \theta/\sin\theta$. This is always the



case so long as $\theta < \frac{1}{2}\pi$. If $\theta =$ or $> \frac{1}{2}\pi$, the section ceases to be that of a hill.

The height of the hill is $2BN = 2r(1-\cos\theta)$. If this $= r$, $\cos\theta = \frac{1}{2}$, $\theta = \frac{1}{3}\pi$. In this case A and O coincide, and $AC = r\sqrt{3}$.

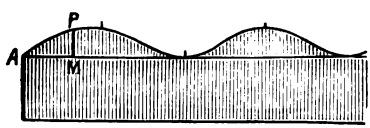
The apparent angular elevation, as seen from C, is the inclination to the horizon of the tangent drawn from C; this is $\text{cosec}^{-1}\sqrt{3}$.

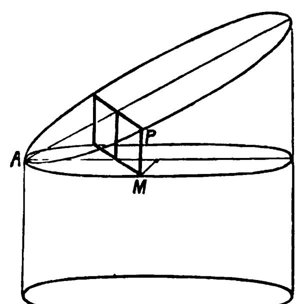
The line joining the tops of equal towers at B and C, which are just mutually visible, is a tangent parallel to BC; therefore the radius to the point of contact makes an angle $\frac{1}{3}\pi$ with the horizon, and the height of either tower is $r(\text{cosec}\,\frac{1}{3}\pi - 1) = \frac{1}{3}r(2\sqrt{3}-3)$.]

12985. (A. S. Eve, M.A.)—A right circular cylinder is cut obliquely and the curved surface is blackened, and the cylinder is then rolled on a plane. Trace the bounding curve of the black area, and find its equation.

Solution by Professors A. Droz-Farny, Krishmachandra De, *and others.*

Let PM be an ordinate, AM an abscissa, and denote them by y, x. Let AM subtend an angle θ at the centre of the circular section, radius r.





Let α be the inclination of the oblique to a circular section.

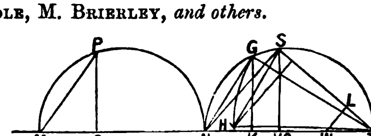
Then $$x = r\theta,$$ $$y = k(1-\cos x/r),$$ in which $$k = \text{T}\tan\alpha.$$

Thus the equation is reducible to the form $m = m\sin x/r$.

12993. (P. W. Flood.)—Given the sum of the squares of the sides containing the vertical angle, and the difference of the segments of the base made by the perpendicular, construct the triangle when the product of the base and square of the perpendicular is a maximum.

Solution by D. Biddle, M. Brierley, *and others.*

Let the square root of the given sum = unity, the given difference = $2d$, the required height = h, the longer of the two sides = x, and the base = $2y$. Then



$$x^2 - h^2 = y^2 + 2dy + d^2, \quad 1 - x^2 - h^2 = y^2 - 2dy + d^2;$$

and we have $\quad 1-2h^2 = 2y^2+2d^2, \quad$ and $\quad 2x^2-1 = 4dy;$
whence $\quad 2h^2y = (1-2d^2)\,y-2y^3, \quad$ and $\quad y = (x^2-\tfrac{1}{2})/(2d).$

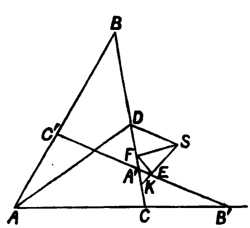
Let
$$z = 2dy = x^2-\tfrac{1}{2},$$
then
$$2h^2y = \frac{(1-2d^2)z}{2d} - \frac{z^3}{4d^3} = \text{a maximum}.$$

Taking the differential coefficients, $3z^2/(4d^3) = (1-2d^2)/(2d)$,
whence
$$z = d\left(\frac{2-4d^2}{3}\right)^{\frac{1}{2}} \text{ and } 2y = \left(\frac{2-4d^2}{3}\right)^{\frac{1}{2}}.$$

The construction is therefore easy. On MN, taken as unity, describe the semicircle MPN. Produce MN to T, making NT = MN, and on NT describe the semicircle NST. Make MP = $2d$; draw PQ perpendicular to MT, and make TV = $\tfrac{1}{2}$QT. Draw VS at right angles, and join ST; then ST = the base of the required triangle. Join SN, make TW = $\tfrac{1}{2}$MP, and draw WL parallel to SN; then TL = z. Wherefore, O being the centre of the semicircle NST, make OK = TL, and draw KG at right angles to NT. Join GN, GT. Finally, from centre T with radius TG, and from centre S with radius = GN, describe arcs intersecting in H. Then HST is the required triangle.

13002. (Professor A. Droz-Farny.)—On considère toutes les transversales Δ qui coupent les côtés BC, AC et AB d'un triangle en A′, B′, C′ de manière à ce que A′B′ = A′C′. L'enveloppe de Δ est une parabole dont on demande la détermination du foyer, de la directrice et la valeur du paramètre.

Solution by G. Heppel, M.A.; Rev. J. L. Kitchin, M.A.; *and others.*



AC, AB are axes, BC is $cx+by-bc = 0$,
B′C′ is $\quad nx+my-mn = 0.$

The condition that A′B′ = A′C′ gives
$$nb = c\,(2b-m).$$

This makes B′C′
$$c\,(2b-m)\,x+mby-mc\,(2b-m) = 0.$$

Differentiating with respect to m, and eliminating m, the envelope is
$$(cx-by)^2-4bc\,(cx+by)+4b^2c^2 = 0,$$
which is a parabola with axis parallel to the median AD, and touching the sides of the triangle.

Hence this construction. Let DS, DA make equal angles with BC. Let DS cut the circumcircle in S. Draw SK parallel to DA, and SF, FE perpendicular to BC, SK. Then S is the focus, and E is the vertex.

[Prof. Droz-Farny sends the following *remarque*:—Le foyer est le point d'intersection de la symédiane correspondant au sommet A avec la circonférence circonscrite au triangle ABC. J'aurais désiré le calcul du paramètre de la parabole en fonction des éléments du triangle.]

13016. (J. J. Walker, F.R.S.)—If α, β, γ, δ are any four vectors, show that $$SV\alpha\beta V\gamma\delta = S\alpha\delta S\beta\gamma - S\gamma\alpha S\beta\delta.$$

Solution by Rev. J. Cullen; G. Heppel, M.A.; *and others.*

This is easily shown by means of the well-known formula

$$V.\beta V\gamma\delta = \delta S\beta\gamma - \gamma S\beta\delta \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

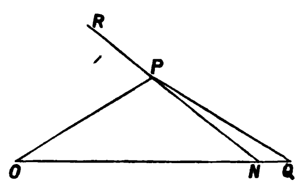
or, as it may be written, $$\beta V\gamma\delta - S\beta V\gamma\delta = \delta S\beta\gamma - \gamma S\beta\delta \quad\ldots\ldots\ldots\ldots\ldots (2).$$

Operate on (2) by $S.\alpha$, and, remembering that $\alpha S\beta V\gamma\delta$ and $S\alpha\beta . V\gamma\delta$ are vectors, we obtain the required result.

[For another proof, see Kelland and Tait's *Quaternions*, 2nd Ed., p. 159, Art. 16.]

13003. (Professor Ramaswami Aiyar, M.A.)—Rays of light proceeding from the centre of the acute-angled hyperbola $x^2/a^2 - y^2/b^2 = 1$ are refracted at the curve, the index of refraction being $\mu = (a^2+b^2)/(a^2-b^2)$. Prove that each refracted ray is equally inclined to the axis with the corresponding incident ray; and the caustic by refraction is the evolute of an hyperbola.

Solution by H. W. Curjel, M.A.; Professor Sanjána; *and others.*



Let O be the centre of the hyperbola, and P any point on it. Draw PQ cutting the axis in Q, and making

$$\angle PQO = \angle QOP.$$

Let the normal PN at P cut the axis in N. Produce NP to R. Let

$$\angle PON = \alpha = \angle PQO, \quad \angle PNO = \beta, \quad \angle NPQ = \phi, \quad \angle OPR = \theta.$$

Then $$\tan\alpha = y'/x', \quad \tan\beta = a^2y'/b^2x',$$

$$\tan\theta = \tan(\alpha+\beta) = \frac{xy(a^2+b^2)}{b^2x^2 - a^2y^2}, \quad \tan\phi = \tan(\beta-\alpha) = \frac{xy(a^2-b^2)}{b^2x^2+a^2y^2};$$

$$\therefore \quad \sin\theta = \frac{a^2+b^2}{a^2-b^2}\sin\phi = \mu\sin\phi;$$

$\therefore$ PQ is the path of the ray OP after refraction.

The equation to PQ is $\xi y' + \eta x' - 2x'y' = 0$, which is normal to the hyperbola $x^2/b^2 - y^2/a^2 = 1$ at the point where it cuts the hyperbola

$$xy(a^2+b^2) = 2abx'y'.$$

Hence the caustic of refraction is the evolute of the hyperbola

$$x^2/b^2 - y^2/a^2 = 1.$$

13014. (EDITOR.)—Solve the equations—

$$x+y+axy = l, \quad y+z+ayz = m, \quad z+x+azx = n.$$

Solution by G. HEPPEL, M.A.; Rev. D. THOMAS, M.A.; *and others.*

Eliminating x and y gives the quadratic

$$a(1+ma)x^2+2(1+ma)x-(l-m+n+aln) = 0.$$

If $1+la = u$, $1+ma = v$, $1+na = w$, $x = a^{-1}(u^{\frac{1}{2}}v^{-\frac{1}{2}}w^{\frac{1}{2}}-1)$, with symmetrical values for y and z.

[Mr. THOMAS gives this Solution :—

Using u, v, w as in the Solution printed, $1+a(x+y)+a^2xy = u$, &c.

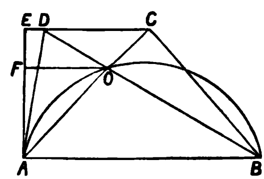
$\therefore\ (1+ax)(1+ay) = u$; $(1+ay)(1+az) = v$; $(1+az)(1+ax) = w$;

hence $(1+ax)(1+ay)(1+az) = (uvw)^{\frac{1}{2}}$, $1+ax = (uvw)^{\frac{1}{2}}+v$,

$$x = a^{-1}(u^{\frac{1}{2}}v^{-\frac{1}{2}}w^{\frac{1}{2}}-1).]$$

13007. (Professor ZERR.)—Construct a trapezoid, given the bases, the perpendicular distance between the bases, and the angle formed by the diagonals.

Solution by H. W. CURJEL, M.A.; Professor NATH COONDOO; *and others.*



Let AB be one of the given bases. On AB bescribe a segment AOB of a circle containing an angle equal to the given angle between the diagonals.

Draw AE perpendicular to AB, and equal to the given distance between the bases. In AE take F so that

AF : FE = AB : other base.

Draw FO parallel to AB, cutting AOB in O. Let BO, AO meet the parallel to AB through E in D and C. Then ABCD is clearly the required trapezoid.

13021. (W. C. STANHAM.)—If the probability of any one aged (t) dying before he is $(t+dt)$ be $at\,dt$, find the average length of life.

Solution by Professor SWAMINATHA AIYAR; *the* PROPOSER; *and others.*

The time (T) lived in the $(n+1)$th interval dt is, on the average,

$$dt\left\{1-adt^2\right\}\left\{1-2adt^2\right\}\dots\left\{1-nadt^2+\tfrac{1}{2}nadt^2\right\};$$

$\therefore\ \log T-\log dt = -\frac{1}{2}an^2dt^2$ in the limit; $\therefore\ T = e^{-\frac{1}{2}an^2dt^2}dt.$

Integrating, from $x = 0$ to $x = \infty$, the expression $e^{-\frac{1}{2}ax^2}dx$, we obtain the average length of life, $(\pi/2a)^{\frac{1}{2}}$.

13008. (Professor GREGG.)—Given two points A and B, and a circle whose centre is O, show that the rectangle contained by OB and the perpendicular from B on the polar of A is equal to the rectangle contained by OB and the perpendicular from A on the polar of B.

Solution by G. HEPPEL, M.A.; H. W. CURJEL, M.A.; *and others.*

Let OA meet the polar CA′E of A in A′, and OB meet the polar FB′D of B in B′. Let BO, CE meet in E, and AO, DF in F.

Draw AD, BC perpendicular to FD, CE.

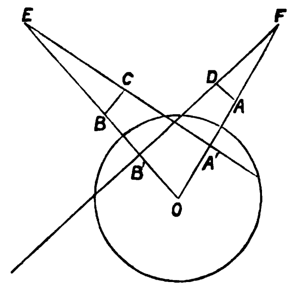
Then triangles OA′E, OB′F are clearly similar, and also triangles A′OB, B′OA.

Hence Δs OAD, OBC are similar;

$\therefore$ OB/BC = OA/AD;

$\therefore$ OB : the perpendicular from B on the polar of A

= OA : perpendicular from A on the polar of B.



12924. (P. W. FLOOD.)—In the figure to the first proposition of the First Book of EUCLID inscribe a circle in the space ABC; and find numerically what part the radius of the required circle is of the given line AB.

Solution by V. J. BOUTON, B.Sc.; Professor RADHAKRISHNAN; *and others.*

If O is the centre of the circle,

r = its radius, and a = AB,

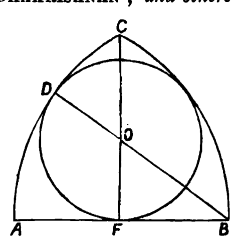
we have BF $= \frac{1}{2}a$, BO $= a-r$, OF $= r$;

hence $r^2+\frac{1}{4}a^2 = (a-r)^2 = a^2-2ar+r^2$,

$$\tfrac{3}{4}a^2-2ar = 0,$$

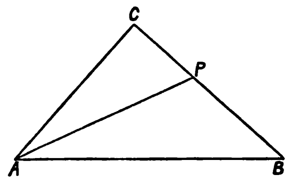
or $$r = \tfrac{3}{8}a;$$

whence construction follows at once.



12922. (M. BRIERLEY.) — Given the hypotenuse of a right-angled triangle, construct it when the product of one of the legs and the line drawn to it which bisects the opposite angle is a maximum.

Solution by V. J. Bouton, B.Sc.; Professor Mukhopadhyay; *and others.*



Let AP bisect the angle A; then

$$CP = \frac{ab}{b+c}, \quad AP^2 = \frac{b^2\{(b+c)^2+a^2\}}{(b+c)^2}.$$

Now $a \cdot AP$ is to be a maximum, c being constant.

$$a^2 \cdot AP^2 = \frac{b^2(c^2-b^2)\{(c^2-b^2)+(b+c)^2\}}{(b+c)^2}$$
$$= b^2(c-b)\,2c.$$

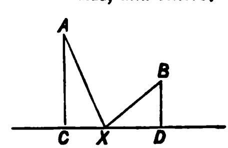
which is to be a maximum; hence, differentiating with respect to b,

$$2bc - 3b^2 = 0, \quad \text{and} \quad b = \tfrac{2}{3}c, \quad a = \tfrac{1}{3}\sqrt{5}\,c.$$

If, then, on c as diameter, we draw a semicircle, and draw $b = \frac{2}{3}c$, we may complete the triangle.

3767. (Editor.)—Two houses stand 750 yards apart on the side of a hill of uniform slope, and at the respective distances of 600 and 150 yards from a brook, which runs in a straight line along the foot of the hill. A man starts from the first house to go to the brook for water, which he is to carry to the second house. Supposing that he can only walk half as fast in going uphill with the water as he can in going downhill to the brook, find the path he must take, and the distance he will have to go, in order to perform his work in the shortest possible time.

Solution by Professors Ramachandra Row, Chakrivarti, *and others.*



Let A, B be the houses, and CD the river. Treating 150 yards as the unit, we have

$$AB = 5, \quad AC = 4, \quad BD = 1, \quad CD = 4.$$

Let the path be AXB; then, since time of descent from one point to another is the least when the path is a straight line, the path consists of two straight lines AX and BX.

Let $CX = x$; then, the time required by the man being irrespective of the inclination of the path to the horizon, provided the path is always uphill or downhill, $AX + 2BX$ is to be a minimum,

i.e., $$(x^2+16)^{\frac{1}{2}} + 2(x^2-8x+17)^{\frac{1}{2}} = \text{a minimum};$$

$$\therefore \quad \frac{x}{(x^2+16)^{\frac{1}{2}}} - \frac{2(x-4)}{(x^2-8x+17)^{\frac{1}{2}}} = 0;$$

$$\therefore \quad 3x^4 - 24x^3 + 111x^2 - 512x + 1024 = 0, \quad \text{and} \quad x = 3\ldots,$$

i.e., he must meet the river at a distance of about 450 yards from C.

[Under Question 7576, solutions of an expanded form of this question are given on pages 82–3 of Vol. xli.]

13022. (P. W. Flood.)—Find x and y when $x^{\frac{1}{2}}+y^{\frac{1}{2}} = x^{\frac{1}{3}}+y^{\frac{1}{3}}$.

Solution by G. Heppel, M.A.; R. F. Davis, M.A.; *and others.*

Let $x = (1+v)^6$, $y = (1-u)^6$; then $u^3-v^3-2(u^2+v^2)+u-v = 0$; and, by putting $u+v = m$, $u-v = n$, we obtain $m^2 = n(2-n)^2/(4-3n)$,

whence
$$x^{\frac{1}{6}} = \tfrac{1}{2}(2-n)\left[1+\{n/(4-3n)\}^{\frac{1}{2}}\right],$$
$$y^{\frac{1}{6}} = \tfrac{1}{2}(2-n)\left[1-\{n/(4-3n)\}^{\frac{1}{2}}\right],$$
and
$$x^{\frac{1}{2}}+y^{\frac{1}{2}} = x^{\frac{1}{3}}+y^{\frac{1}{3}} = (2-n)^3/(4-3n),$$
where n is arbitrary and less than 1.

12991. (M. Brierley.)—Construct a triangle such that the product of the three sides shall be equal to four times the cube of the perpendicular from the vertical angle.

Solution by D. Biddle, Prof. Radhakrishnan, *and others.*

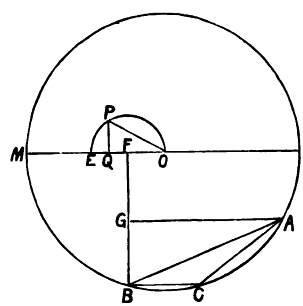
Here, $abc = 4h^3 = 4R\Delta = 2Rha$;

$\therefore\ h^2 = \frac{1}{2}Ra$.

Several triangles will fulfil the conditions.

Let O be the circumcentre, and OM = R. Bisect OM in E, and OE in F; also draw a semicircle on OE. Take any point P on the circumference of this semicircle, and draw PQ perpendicular to OE.

Then OP, OQ are respectively the height and base of a triangle fulfilling the conditions. The simplest case (as in the cut) is when each = OE.

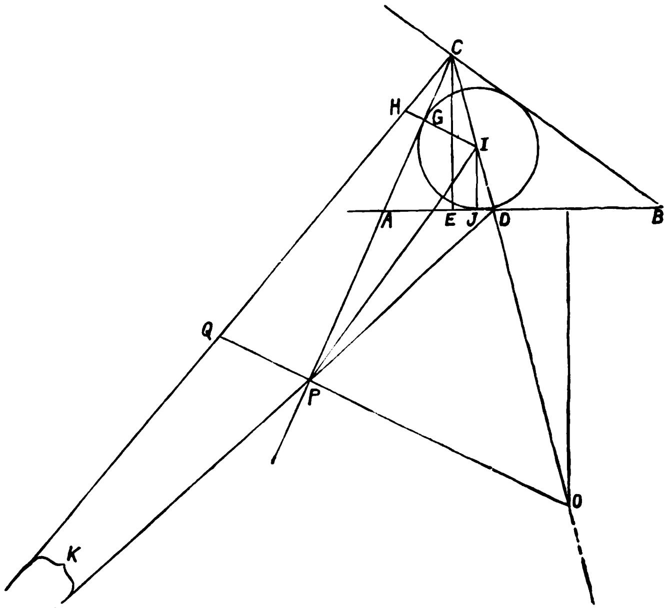


482. (J. H. Swale.)—In the triangle ABC, take I and O, the centres of the inscribed and escribed circles, the latter touching the sides CA, CB produced. Draw OP perpendicular to CA; also CQ parallel to PI, meeting OP in Q. Then 2PQ = perpendicular CE (upon AB). Also, if CO cut AB in D, and DP be drawn to meet CQ in K, then QK = QC.

Solution by M. Brierley, Prof. Gopalachanar, *and others.*

Draw the radius IG perpendicular to CA, and produce it to meet CK in H. Then (1) IG . PC = $\frac{1}{2}$CE . AB. But PG = AB, and, by similar

triangles, IH . PG = PC . IG ; $\therefore$ IH = PQ = $\frac{1}{2}$CE, *i.e.*, 2PQ = CE.



(2) IP : CK = DI : DC = IP : CK = IJ (= IG) : CE = QH : CK. Again, by similar triangles, IH : CQ :: CE : CK; $\therefore$ 2QC = CK, *i.e.*, QK = QC. [For another Solution, see Vol. LXI., p. 72.]

12947. (F. G. Taylor, M.A., B.Sc.)—Prove that

$$\cosh x = \cos x\left\{1 - \frac{2^2x^4}{4!} + \frac{2^4x^8}{8!} - \dots\right\} + x\sin x\left\{\frac{1}{1!} - \frac{2^2x^4}{5!} + \frac{2^4x^8}{9!} - \dots\right\}$$
$$+ 2x^3\sin x\left\{\frac{1}{3!} - \frac{2^2x^4}{7!} + \frac{2^4x^8}{11!} - \dots\right\};$$

$$\sinh x = x\cos x\left\{\frac{1}{1!} - \frac{2^2x^4}{5!} + \frac{2^4x^8}{9!} - \dots\right\} + 2x^2\sin x\left\{\frac{1}{2!} - \frac{2^2x^4}{6!} + \frac{2^4x^8}{10!} - \dots\right\}$$
$$- 2x^3\cos x\left\{\frac{1}{3!} - \frac{2^2x^4}{7!} + \frac{2^4x^8}{11!} - \dots\right\}.$$

Solution by H. W. CURJEL, M.A.; *the* PROPOSER; *and others.*

By TAYLOR's theorem, we have

$$e^{x+h}\sin(x+h) = e^x\sin x + h\,.\,2^{\frac{1}{2}}\,.\,e^x\sin(x+\tfrac{1}{4}\pi)+\dots;$$

$$\therefore\quad e^h\sin(x+h) = \sin x + h\,.\,2^{\frac{1}{2}}\sin(x+\tfrac{1}{4}\pi)+\frac{h^2}{2!}\,2^{\frac{2}{2}}\sin\left(x+\frac{2\pi}{4}\right)+\dots$$

Put $x = \frac{1}{2}\pi - h$, and change the sign of h; then

$$e^h = \cos h + h\,.\,2^{\frac{1}{2}}\cos(h-\tfrac{1}{4}\pi)+\frac{h^2}{2!}\,2^{\frac{2}{2}}\cos\left(h-\frac{2\pi}{4}\right)+\dots;$$

$$e^{-h} = \cos h - h\,.\,2^{\frac{1}{2}}\cos(h+\tfrac{1}{4}\pi)+\frac{h^2}{2}\,2^{\frac{2}{2}}\cos\left(h+\frac{2\pi}{4}\right)-\dots;$$

whence, by addition and halving, changing h into x,

$$\cosh x = \cos x + x\,.\,2^{\frac{1}{2}}\sin x\sin\tfrac{1}{4}\pi+\frac{x^2}{2!}\,2^{\frac{2}{2}}\cos x\cos\frac{2\pi}{4}+\frac{x^3}{3!}\,2^{\frac{3}{2}}\sin x\sin\frac{3\pi}{4}$$

$+\dots =$ &c. Similarly, by subtraction, we obtain the second identity.

10235. (EDITOR.)—If a, b, c be the sides of a triangle, p_1, p_2, p_3 the perpendiculars thereon from the opposite corners, and Δ the area, solve the equation $a(p_1^2-x^2)+b(p_2^2-x^2)+c(p_3^2-x^2) = 2\Delta$.

Solution by H. J. WOODALL, A.R.C.S.; Prof. GOPALACHANAR; *and others.*

To make equation homogeneous, multiply right-hand side by $2\Delta/k$.

Then
$$x^2 = \left\{ap_1^2+bp_2^2+cp_3^2-(2\Delta)^2/k\right\}/(a+b+c)$$
$$= (2\Delta)^2\left\{1/a+1/b+1/c-1/k\right\}/(a+b+c),$$

since
$$ap_1 = bp_2 = cp_3 = 2\Delta.$$

7316. (Professor ORCHARD, M.A.)—Supposing it possible for the earth to collide with an asteroid, the whole of whose kinetic energy was thus turned into heat, find the heat which might be produced by an asteroid of the mass of 1000 kilogrammes reaching the earth's surface with "the velocity from infinity."

Solution by H. J. WOODALL, A.R.C.S., Prof. CHAKRIVARTI, *and others.*

The kinetic energy of the mass is

$$\tfrac{1}{2}mV^2 = \tfrac{1}{3}(4\pi\rho R^2 m) = \tfrac{4}{3}\times 3{\cdot}1416\times 5{\cdot}56\times(6{\cdot}36\times10^8)^2\times1000\times1000$$
$$= 942\times10^{22},\ \text{about.}$$

1 gramme of water would be heated 1° C. by the amount of work
$$= 4\cdot 2 \times 10^7 \text{ ergs}.$$
Therefore number of thermal units
$$= 942 \times 10^{22}/4\cdot 2 \times 10^7 = 2\cdot 3 \times 10^{17}, \text{ about}.$$

9801. (W. J. C. SHARP, M.A.)—If P_r denote the Legendre's coefficient of the r^{th} order of $\frac{1}{2}(k+1/k)$, show that
$$\int_0^x \frac{dx}{\{(1-x^2)(1-k^2x^2)\}^{\frac{1}{2}}} = x + P_1\frac{kx^3}{3} + P_2\frac{k^2x^5}{5} + \dots + P_r\frac{k^r x^{2r+1}}{2r+1} + \&c.$$

Solution by H. J. WOODALL, A.R.C.S.; Prof. GOPALACHANAR; *and others.*

We have, CARR, *Synopsis*, § 2936 (slightly altering the notation),
$$(1-2lx^2+x^4)^{-\frac{1}{2}} = 1 + X_1x^2 + X_2x^4 + \dots + X_rx^{2r} + \dots;$$
$$\therefore \quad \{1-x^2(1+k^2)+k^2x^4\}^{-\frac{1}{2}} = 1 + P_1kx^2 + P_2k^2x^4 + \dots + P_rk^rx^{2r} + \dots,$$
where P_r has the signification given in the Question [since l is replaced by $\frac{1}{2}(k+1/k)$]. Then integrate and we find as above.

9782. (W. J. C. SHARP, M.A.)—If ${}_nP_r$ denote the coefficient of x_r in the expansion of $(1+x)^n$, &c., ${}_nC_r$ denote the number of combinations of n things taken r together, form the equations of differences which determine ${}_nP_r$, and ${}_nC_r$, and hence show that these are equal.

Solution by H. J. WOODALL, A.R.C.S.; Prof. SARKAR; *and others.*

We have $\quad (1+x)^n = 1 + {}_nP_1x + \dots + {}_nP_{r-1}x^{r-1} + {}_nP_rx^r + \dots,$
$$(1+x)^{n+1} = 1 + {}_{n+1}P_1x + \dots + {}_{n+1}P_{r-1}x^{r-1} + {}_{n+1}P_rx^r + \dots;$$
but
$$(1+x)^{n+1} = (1+x)(1+x)^n;$$
therefore, by equating coefficients of x^r, ${}_{n+1}P_r = {}_nP_r + {}_nP_{r-1}$,

whence $\Delta_n {}_nP_r = {}_nP_{r-1}$ (Δ refers to n only). Again, number of combinations of $n+1$ things r together $= {}_{n+1}C_r$; these are made up of ${}_nC_{r-1}$, where a certain thing always occurs together with ${}_nC_r$ where this does not occur (or, as we may put it, to get ${}_{n+1}C_r$ we must add ${}_nC_{r-1}$ to ${}_nC_r$)
therefore
$$\Delta_n {}_nC_r = {}_nC_{r-1},$$
But this equation is the same as $\Delta_n {}_nP_r = {}_nP_{r-1}$; therefore ${}_nC_r$ and ${}_nP_r$ are equal, since ${}_nC_1 = n = {}_nP_1$.

12898. (H. Fortey.)—**Four random chords are drawn in a circle; find the chance of any number of intersections from 0 to 6, both included.**

Note by the Editor.

1. This Question arose out of a very old one (Question 3631), proposed by me many years ago, and of which a Solution has been given by Mr. Fortey on page 59 of Vol. LXIV. But the Solution of the present Question (12898) has led to so much prolonged controversy that we summarize the whole below.

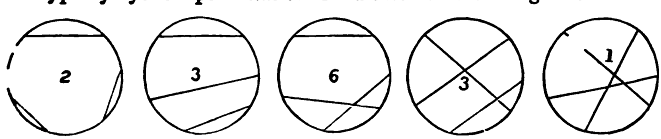
2. Mr. Biddle solved the Question as follows:—

The odds against intersection of any two random chords are 2 to 1; for $2\int_0^1 x(1-x)\,dx = \frac{1}{3}$, whilst $\int_0^1 \{(1-x)^2+x^2\}\,dx = \frac{2}{3}$. And it is easy to see that the several probabilities required are given by the successive terms in the expansion of $(\frac{2}{3}+\frac{1}{3})^{\frac{1}{2}n(n-1)}$, where n = the number of chords concerned; in the present case, $(\frac{2}{3}+\frac{1}{3})^6$.

3. Mr. Fortey having seen the solution by Mr. Biddle, remarks that his own solution gives different results, even when there are only 3 random chords; hence, confining his demonstration to that case, he leaves mathematicians to decide which is right.

He gives, as his definition, that a random chord is the join of two random points on the circumference.

Take 6 random points. These can be joined by 15 sets of 3 random chords, all of which sets are equally probable. These 15 sets are included in 5 types figured below, and the number of sets derived from each type by cyclical permutation is indicated on each diagram.



Therefore, if I^n be the chance of n intersections, we have

$$I_0 = \frac{2+3}{15} = \frac{1}{3},\ I_1 = \frac{2}{5},\ I_2 = \frac{1}{5},\ I_3 = \frac{1}{15}$$

(see also paragraph 7 of Solution of Question 3631).

Similarly he finds for 4 random chords

$$I_0 = \frac{2}{15},\ I_1 = I_2 = \frac{4}{15},\ I_3 = \frac{4}{21},\ I_4 = \frac{2}{21},\ I_5 = \frac{4}{105},\ I_6 = \frac{1}{105}.$$

4. Mr. Simmons writes as follows:—"The Editor having desired me to comment on the above, I am the more willing to comply as the point at issue is both interesting and important. Mr. Fortey's method of marking all the random points first, and then joining them afterwards, is very ingenious, and, granting his own definition, undoubtedly legitimate. Presuming that definition, I have worked out the three-chord problem by joining each pair of points before marking the next pair, and then applying double integration; and, as our two independent methods lead to the same numerical results, $\frac{1}{3}$, $\frac{2}{5}$, $\frac{1}{5}$, $\frac{1}{15}$, I have no hesitation in affirming Mr. Fortey's solution, for the three-chord problem at least, to be correct.

"Mr. BIDDLE presumes the same definition if, as I suppose, his x denotes length of arc of a circle whose circumference is unity. His deduction of $\frac{1}{3}$ as the chance of intersection of any two random chords is also correct. But his conclusion in the next sentence is not at all 'easy to see.' On the contrary, I hold it to be entirely erroneous, the binomial formula being here quite inapplicable. In order to perceive clearly where the fallacy lies, let us confine ourselves to the chance of *three* intersections of *three* random chords A, B, C. The intersections of A and B, B and C, C and A are three distinct events, which we will call P, Q, R. The chance of any one of them, considered alone, is $\frac{1}{3}$. Therefore, according to Mr. BIDDLE, the chance of their joint occurrence is $\frac{1}{3}.\frac{1}{3}.\frac{1}{3}$, or $\frac{1}{27}$. But this is not true, for the simple reason that the *three events are not independent*.

"In fact, the matter stands as follows:—Any two of the events P and Q are relatively independent, so long as the third event R is ignored. The chance of the joint occurrence PQ is therefore $\frac{1}{3}.\frac{1}{3}$. But, this joint occurrence being presumed, the probability of R is no longer $\frac{1}{3}$, as it was at first. Our knowledge of the fact that A and C both intersect B limits the range of possible positions of A and C relatively to B, *and therefore their range of possible positions relatively to one another*. Consequently, to find the chance of P, Q, R all happening together, we must not multiply $\frac{1}{3}.\frac{1}{3}$ by $\frac{1}{3}$, for $\frac{1}{3}$ is the chance of intersection of two random chords which, in relation to one another, are absolutely independent. On the contrary, the chance required will be $\frac{1}{3}.\frac{1}{3}.\rho$, where ρ denotes the chance that two chords, drawn originally at random, *but now known both to intersect the same chord*, intersect one another.

"Now I can quite imagine Mr. BIDDLE arguing: 'R being absolutely independent of either P or Q taken separately, and P being absolutely independent of Q, how in the world can the probability of R be affected by the joint occurrence of P and Q?' To which I reply that, curious and self-contradictory though it may seem, it is a fact. And, if any one is disposed to question it, I shall be willing, with the EDITOR's permission, to discuss it further.

"Moreover, according to Mr. FORTEY, $\frac{1}{3}.\frac{1}{3}.\rho$ is equal to $\frac{1}{15}$, so that $\rho = \frac{3}{5}$, an interesting result which might well be proposed anew for independent solution: 'Three random chords (defined as above) being drawn in a circle, and it being found that two of them intersect the third, prove that the odds are 3 to 2 in favour of their intersecting one another.' To which, with reference to Mr. FORTEY's result, $I_0 = \frac{1}{3}$, might also be added: 'Three random chords being drawn in a circle, and it being known that one of them is not intersected by either of the other two, prove that the odds are 3 to 1 against the mutual intersection of these two latter.'"

5. Mr. BIDDLE rejoins as follows:—"I agree that Mr. FORTEY's solution is correct if the six points (in the case of three chords) be given before any junctions are made. For, to confine ourselves, as Mr. SIMMONS has done, to the probability of all three chords intersecting with each other, let A, B, C, D, E, F (Fig. 1) be the six points in order, and A be that from which the first junction is made. AD, BE, CF are the necessary junctions; the probability of the first is $\frac{1}{5}$, of the second $\frac{1}{3}$, and of the third unity; there being no alternative. Consequently, we have

$$P = \tfrac{1}{5}.\tfrac{1}{3}.1 = \tfrac{1}{15}.$$

"In this case, after selection of the starting-point, junction is always effected with the middle one (if more than one) of the remaining points, B, C, *D*, E, F; C, *E*, F. But it is by no means clear that all arrangements of five points about A (fixed) are equally likely; for, if the circumference be divided into five equal portions, it seems less likely that the five points should fall in a single particular compartment than be spread over the whole; and, as AD is deflected more and more from the diameter, it becomes less and less likely to have two random points out of four on either side of it. Thus, in arranging five points on the circumference in every possible position about A, we find that the middle one, D, varies in position less freely than the others, that is to say, its velocity of transition is slower. Consequently, a chord drawn from A with equal probability to any other point on the circumference is less likely to terminate near the middle one of five other points chosen at random than near one of the side ones. To explain—let the circumference be divided into five equal portions, *b*, *c*, *d*, *e*, *f* (Fig. 2), A being at the junction of *b* and *f*. There is not one of these divisions into which B, C, D, E, F severally enter not; but the interchange is not equal. B may be in *f* as often as F is in *b*, but D will not be in *b* or *f* so often as B or F is in *d*; nor will *D* be in *b* or *f* so often as it is in *c* or *e*, nor in one of these so often as in *d*. Wherefore the junction of AD, as representing a random chord, is not equally probable for all arrangements of the five points about A. And the same may be said of BE.

"Consequently, the taking of six random points all at once as the terminals of three random chords, and then regarding all possible junctions of those six points in pairs as equally probable, is misleading. Mr. Simmons says that he has proved Mr. Fortey's method sound by finding it agree with his own results produced by *double* integration. So far as I can judge, nothing short of quintuple integration will suffice, even in the case of three chords, to solve in that way.

"In conclusion, I would beg to make the following statements:—

"(*a*) The chords are completely independent of each other; the position of one does not in the least interfere with that of another.

"(*b*) The intersections also are independent of each other, to the extent that no two occur together of necessity. When only three chords are concerned, there is not a point on the circumference from which the third chord cannot be drawn so as to cut one or other, or both, or neither, of the other two, at will, when the two already intersect. But, of course, the probabilities vary, not being always identical, as in the casting of dice. The *one-third* chance of intersection of two random chords is the *average* of chances that differ widely, and the same may be said of three or more chords.

"(*c*) It is immaterial in what order the chords are drawn, but in the case before us let A, B, C be the first, second, and third in every trial. When a vast number of circles is subsequently inspected, A, B, C will be found to have fared pretty much alike.

"(*d*) Let P, Q, R be intersections, respectively, of A and B, B and C, C and A, and let *p*, *q*, *r* represent non-intersections of the same. Then, if $3n$ represent an immense number of trials in different circles, there will be nP to $2np$, nQ to $2nq$, nR to $2nr$.

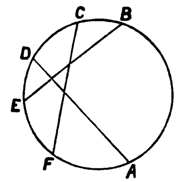


Fig. 1.

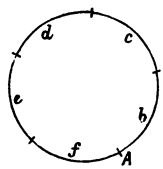


Fig. 2.

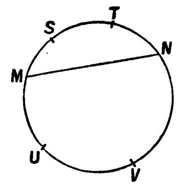


Fig. 3.

"(e) The concurrences of P with Q, of Q with R, of R with P, will be equal; likewise of P with *q*, Q with *r*, R with *p*; also of P with *r*, Q with *p*, R with *q*; and the two last sets of equalities will be found identical.

"(*f*) I beg to submit, therefore, the following scheme, as carrying out these concurrences :—

P P P P P P P P P *p p p p p p p p p p p p p p p p p p*
Q Q Q *q q q q q q* Q Q Q Q Q Q *q q q q q q q q q q q q*
R *r r* R R *r r r r* R R *r r r r* R R R R *r r r r r r r r*;

in which we have the number of trials reduced to 27, in order to give all possible concurrences in their average relation as regards frequency. P, Q, R occur nine times each; *p*, *q*, *r* occur eighteen times each; *pqr* occurs eight times, *pq*R four times, *p*Q*r* four times, P*qr* four times, *p*QR twice, P*q*R twice, PQ*r* twice, and PQR once.

"(*g*) It might further be urged that the number of terms in the expansion given by me exactly fulfils the requirements of the question as to number of intersections, whereas the other method of solution is involved in ever-increasing difficulty.

"(*h*) But, in order to show clearly that the method of choosing the extremities of all the chords before drawing any is false, it may suffice to draw attention to the diagram (Fig. 3) in which the case supposed by Mr. Simmons is portrayed. MN is the chord known to be intersected by the other two. Then S, T on one side, and U, V on the other, being chosen at random, we can join them through MN in two ways, and in two ways only, US, VT or UT, VS (the latter resulting in intersection of the new chords, the former not), and these, according to Mr. Fortey's method, would appear to have an equal chance; but Mr. Simmons says the respective chances are as 2 : 3. The result is clearly unaffected by the position of MN, or of S, T and U, V on their respective sides of it."

The subject is well worthy of further discussion.

6. Mr. Simmons finally remarks as follows :—

"Mr. Biddle's rejoinder, given above, seems to me vague and inconclusive. A careful study of the whole problem afresh has convinced me more than ever that Mr. Fortey's solution is correct. To be quite clear, let us suppose three chords determined by six random points located one after another in the following order of *time*, viz., O, X, Y, Z, U, V, the circle being turned round so that O is always the lowest point.

"I propose to give *four* new proofs, each of which will show the chance of three intersections to be $\frac{1}{15}$, and not $\frac{1}{27}$.

"(i.) Mark off the whole six points, and afterwards join them two and

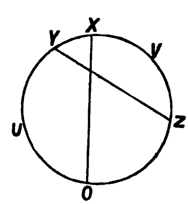
two at random. Here, out of $\dfrac{6\,.\,5}{1\,.\,2}$ or 15 possible ways of joining, only one will give three intersections, *i.e.*, the case where each point is joined with its opposite. The chance of three intersections is therefore $\frac{1}{15}$; which was required.

"Mr. BIDDLE objects that in the above argument 'it is by no means clear that all arrangements of five points about O are equally likely.' To me it seems self-evident, but I will give a proof. Let the left-hand arc OX $= x$, and the right-hand arc OX $= 1-x$. Then the chance that Y, Z, U, V lie equally on both sides of X is $6\int_0^1 x^2(1-x)^2\,dx = \frac{1}{5}$. The chance for only *one* on the left of X is $4\int_0^1 x(1-x)^3\,dx = \frac{1}{5}$, for *three* on the left is $4\int_0^1 x^3(1-x)\,dx = \frac{1}{5}$, for *none* on the left is $\int_0^1 (1-x)^4\,dx = \frac{1}{5}$, and for *all* on the left is $\int_0^1 x^4\,dx = \frac{1}{5}$. Thus, relatively to the other points, X is just as likely to be in any one position as in any other; and similarly for Y, or Z, or U, or V. The above proof is therefore strictly legitimate; but we will proceed.

"(ii.) Join the first chord OX before marking the other points Y, Z, U, V; these latter being then joined at random. Here the chance that Y, Z, U, V fall two and two on opposite sides of OX is, as above, $\frac{1}{5}$. Joining them with one another, the chance that any point Y becomes joined to its remotest point is $\frac{1}{3}$, in which case the remaining two points also are oppositely situated; and the chance required is $\frac{1}{5}\,.\,\frac{1}{3} = \frac{1}{15}$, as before. Mr. BIDDLE may possibly object to this proof also, so we will devise another.

"(iii.) Surely it is obvious that, *a circular circumference being divided by $n+1$ random points, the $(n+1)$th point is equally likely to fall in any one of the n arcs obtained by the previous n points.* This being presumed, take O, X, and join them. The third point Y will give three arcs on each of which the next point Z being equally likely to fall, its chance of falling on the arc opposite to Y is $\frac{1}{3}$. Joining YZ, and marking the fifth point U, we obtain five arcs on each of which the last point V is equally likely to fall; its chance of appearing on the arc most remote from U is therefore $\frac{1}{5}$, giving $\frac{1}{3}\,.\,\frac{1}{5}$, or $\frac{1}{15}$, as the required probability. The statement in italics, which, to my mind, is axiomatic, may, however, be questioned by somebody; so we will give a final proof.

"(iv.) Mark off O, X, and join them; then Y, Z, and join them; lastly U, V. Starting from O in the direction of the hands of a clock, denote the lengths of the arcs OX, OY, OZ, OU, OV by x, y, z, u, v. Then the chance that Y occurs to the left of X, and Z to the right, and U between O and Y, and V between X and Z, as in the figure,



is $$\int_0^1\int_0^x\int_x^1\int_0^y\int_x^z dx\,dy\,dz\,du\,dv = \tfrac{1}{120}.$$

Doubling this to allow for interchange of Y and Z, and again for interchange of U and V, we obtain $\frac{1}{30}$.

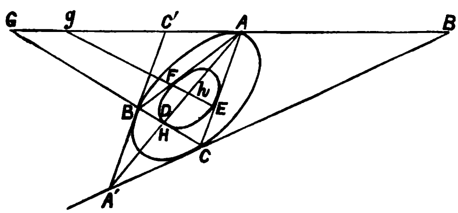
"These are all the cases, and the only cases, in which three intersections can occur. The required chance is therefore $\frac{1}{30}+\frac{1}{30}=\frac{1}{15}$.

"(v.) I had a fifth proof, by double integration, which has got lost, but sufficient space has been already occupied. Of the above solutions, my favourite is No. (iii.), and, adding Mr. Fortey's, there are five in all. Now Mr. Biddle must surely admit that it is curious (to say the least of it) that all these solutions should arrive at the same result. Whatever objections may be adduced against the others, he will not venture to question the conclusiveness of the last. The foregoing arguments will, I trust, justify me in asserting that the correct answer to this most interesting question is incontestably, and beyond all possibility of doubt, $\frac{1}{15}$, and not $\frac{1}{27}$."

7. Mr. Biddle adds that he finds the integration in the solution (iv.) quite correct; and he would be inclined to accept it as conclusive if assured of its adequacy, which he is not at present disposed to admit.

13015. (R. Lachlan, Sc.D.)—A triangle ABC is inscribed in a conic, and the tangents at A, B, C form the triangle A′B′C′. Show that the pole of B′C′ with respect to any conic inscribed in the triangle ABC lies on the straight line AA′.

Solution by G. Heppel, M.A.; H. W. Curjel, M.A.; *and others.*



Let BC cut B′C′ in G, and AA′ in H.

Then G is the pole of AA′ with respect to the conic ABC.

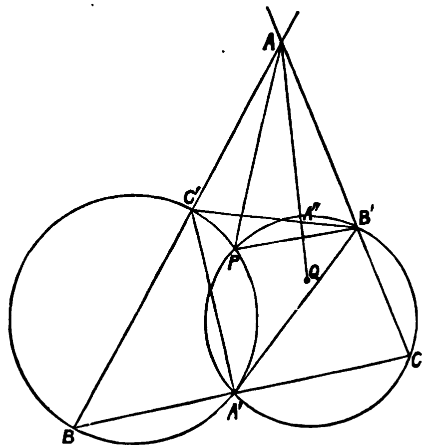
Therefore the range (CB, HG) is harmonic.

Inscribe in $\triangle$ABC a conic DEF touching BC, CA, AB in D, E, F.

Let EF cut AA′ in h, and B′C′ in g. Then (EF, hg) is harmonic, and the polar of g with respect to DEF passes through A, for A is the pole of EF. Therefore AA′ is the polar of g with respect to DEF. Therefore h is the pole of B′C′ with respect to DEF.

13006. (Professor Morley.)—Let ξ_1, ξ_2, ξ_3 be the vertices, and x_1, x_2, x_3 the sides, of one triangle; and let η_1, η_2, η_3 and y_1, y_2, y_3 be the vertices and sides of a second triangle. If lines through ξ_1, ξ_2, ξ_3, making a given angle α with y_1, y_2, y_3, respectively, meet at a point, prove that lines through η_1, η_2, η_3, making the opposite angle $-\alpha$ with x_1, x_2, x_3, respectively, meet at a point. Apply this to the case when η_2 coincides with ξ_1, η_3 with ξ_2, η_1 with ξ_3.

Solution by R. F. DAVIS, M.A.; Professor MUKHOPADHYAY; *and others.*



Let ABC be a given triangle, A′, B′, C′ points lying respectively on the sides BC, CA, AB. Then, by a well-known theorem, the circumcircles of the triangles AB′C′, BC′A′, CA′B′ cointersect in a point P; and, consequently, the angles CA′P, AB′P, BC′P are equal ($=\alpha$, suppose). Let Q be the "inverse point" to P with respect to the triangle ABC, so that AP, AQ are "isogonal conjugates" with respect to AB, AC, &c.

Join AQ, and let it meet B′C′ in A″.

Then the angle AA″C′

$$= AB'C' + QAC = AB'C' + PAC' = AB'C' + PB'C' = AB'P = \alpha.$$

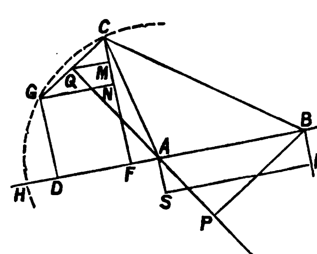
Similarly, BQ, CQ make with C′A′, A′B′ each an angle $=\alpha$.

Conversely, changing the notation, &c.

The particular case of the question deduces the existence of a second Brocard point from that of the first. [The case when $\alpha = \frac{1}{2}\pi$ is given in SALMON'S *Conics*, § 54, Ex. 7.]

12972. (Professor KRISHNACHANDRA DE, M.A.)—Given two fixed points: draw a straight line through a third given point so that the rectangle contained by the perpendiculars drawn upon it from the first two given points may be equal to a given square.

Solution by I. ARNOLD, Professor RADHAKRISHNAN, *and others.*



Let B and C be the two fixed points and A the third given point. Join the three points. Produce BA to H, and let fall the perpendicular CF. To AB apply a rectangle equal to the given square. Make FD = 2BR, one of the sides of this rectangle. From A as centre, with AC as radius, describe the circle CGH. Draw DG parallel to CF, and cutting the circle in G. Join GC and bisect in Q. Join QA and

produce, and let fall the perpendicular BP on QA produced. QAP is the line required.

Draw QM and GN parallel to DF.

Now $$QM = \tfrac{1}{2}GN = \tfrac{1}{2}DF = BR.$$

The triangles CQM and ABP are right-angled and similar;

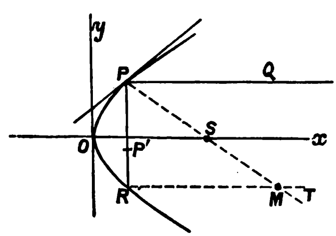
$$\therefore \quad AB \times QM = BP \times CQ,$$

or $$AB \times BR = BP \times CQ = \text{given square}; \quad \therefore \quad \&c.$$

When FD is greater than FH, the problem is impossible.

12989. (R. Knowles, B.A.) — From a point P in the parabola $y^2 = 4ax$, chords PQ, PR are drawn at right angles: show that, as P moves on the curve, the locus of the intersection of QR with the normal at P is the parabola $y^2 = 4a(x-4a)$.

Solution by Professors Cochez, A. Droz-Farny, *and others.*



P décrit la parabole $y^2 = 4ax$.

Prenons la position initiale de P telle que PQ soit parallèle et PR perpendiculaire à l'axe; alors l'hypoténuse du triangle rectangle PQR devient parallèle également à l'axe Ox et se confond avec la droite RT. La normale PS en P rencontre l'hypoténuse en M; c'est le point de Frégier. La sous-normale $P'S = 2a$; par suite $RM = 4a$ et les coordonnées de M sont

$$x = x_1 + 4a, \quad y = -y_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1, 2),$$

x_1 et y_1 étant celles de P.

De plus $$y_1^2 = 4ax_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4).$$

Eliminant x_1 et y_1 entre ces trois relations, on a pour l'équation du lieu

$$y^2 = 4a(x-4a).$$

La même question pour l'ellipse donne pour l'équation du lieu

$$\frac{c^2x^2}{a^2(a^2+c^2)^2} - \frac{c^2y^2}{b^2(a^2+c^2)^2} = 1.$$

La même question pour l'hyperbole donne pour l'équation du lieu

$$\frac{c^2x^2}{a^2(a^2+c^2)^2} - \frac{c^2y^2}{b^2(a^2+c^2)^2} = 1,$$

en posant dans les deux cas $c^2 = a^2 - b^2$.

13013. (Professor Cochez.)—On donne la courbe

$$y^3 - x^2 = 0 \quad \text{et la droite} \quad ux + vy - 1 = 0 \;\ldots\ldots\ldots\; (1, 2).$$

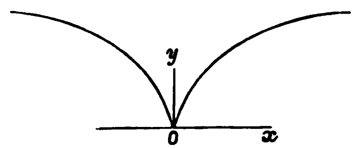
(1) Construire la courbe. (2) A quelles conditions doivent être assujetties u et v pour que deux des points soient à égale distance du troisième? Ces conditions étant remplies, (3) trouver l'enveloppe de la droite (2).

Solution by H. W. Curjel, M.A.; Rev. J. L. Kitchin, M.A.; *and others.*

Take the straight line

$$\frac{x - b^{\frac{3}{2}}}{\cos\theta} = \frac{y - b}{\sin\theta} = r,$$

passing through point P $(b^{\frac{3}{2}}, b)$ on the curve.



Where it cuts the curve,

$$r^3\sin^3\theta + 3br^2\sin^2\theta + 3b^2r\sin\theta + b^3 - r^2\cos^2\theta - 2b^{\frac{3}{2}}r\cos\theta - b^3 = 0.$$

Hence, if the other two points of intersection are equidistant from P,

$$3b\sin^2\theta = \cos^2\theta\,; \quad \therefore \quad b = 1/(3\tan^2\theta)$$

$$\therefore \quad u = -\frac{3\sqrt{3}\tan^3\theta}{\sqrt{3}-1} = -\frac{3\sqrt{3}\,k^3}{\sqrt{3}-1}, \qquad v = \frac{3\sqrt{3}\tan^2\theta}{\sqrt{3}-1} = \frac{3\sqrt{3}\,k^2}{\sqrt{3}-1}\,;$$

$$\therefore \quad \text{envelope is } -\frac{8y^3}{27x^2} + \frac{4y^3}{9x^2} = \frac{\sqrt{3}-1}{3\sqrt{3}}, \quad \text{or} \quad 4y^3 = (9 - 3\sqrt{3})\,x^2,$$

another semicubical parabola.

The curve is as in the figure.

13056. (J. J. Walker, F.R.S.)—Prove that, if α, β, γ, δ are any four vectors,

$$2\mathrm{V}\alpha\beta\gamma\delta = \mathrm{V}\alpha\mathrm{V}\beta\gamma\delta - \mathrm{V}\beta\mathrm{V}\delta\gamma\alpha + \mathrm{V}\gamma\mathrm{V}\alpha\beta\delta - \mathrm{V}\delta\mathrm{V}\gamma\beta\alpha,$$

pointing out a rule for forming the succeeding terms from the preceding.

Solution by the Proposer; G. Heppel, M.A.; *and others.*

$$\begin{aligned}\mathrm{V}\alpha\beta\gamma\delta &= \alpha\mathrm{S}\beta\gamma\delta + \mathrm{V}\alpha\mathrm{V}\beta\gamma\delta \\ &= \mathrm{V}\gamma\delta\,\mathrm{S}\alpha\beta + \mathrm{V}\delta\beta\,\mathrm{S}\gamma\alpha + \mathrm{V}\beta\gamma\,\mathrm{S}\alpha\delta \quad (\text{Tait, } Q., \S\,92) \\ &\qquad + \mathrm{V}\alpha\beta\,\mathrm{S}\gamma\delta + \mathrm{V}\gamma\alpha\,\mathrm{S}\beta\delta + \mathrm{V}\alpha\delta\,\mathrm{S}\beta\gamma \quad (ib.\ \S\,90) \;\ldots\ldots\ldots\; (1);\end{aligned}$$

which is, Mr. Walker believes, a new and useful formula.

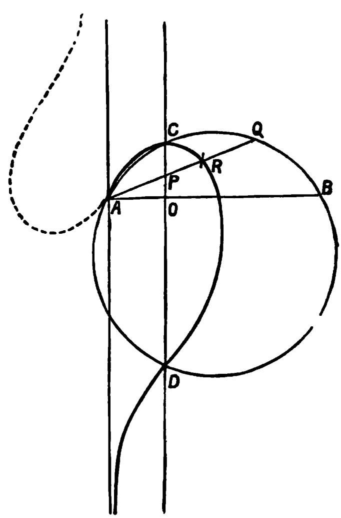
But

$$\begin{aligned}-\mathrm{V}.\beta\mathrm{V}\delta\gamma\alpha &= \mathrm{V}\delta\beta\,\mathrm{S}\gamma\alpha + \mathrm{V}\beta\gamma\,\mathrm{S}\alpha\delta + \mathrm{V}\alpha\beta\,\mathrm{S}\gamma\delta, \\ +\mathrm{V}.\gamma\mathrm{V}\alpha\beta\delta &= \mathrm{V}\gamma\alpha\,\mathrm{S}\beta\delta + \mathrm{V}\beta\gamma\,\mathrm{S}\alpha\delta + \mathrm{V}\gamma\delta\,\mathrm{S}\alpha\beta, \\ -\mathrm{V}.\delta\mathrm{V}\alpha\beta\gamma &= \mathrm{V}\alpha\delta\,\mathrm{S}\beta\gamma + \mathrm{V}\delta\beta\,\mathrm{S}\gamma\alpha + \mathrm{V}\gamma\delta\,\mathrm{S}\alpha\beta\,;\end{aligned}$$

adding these three equalities to $V\alpha V\beta\gamma\delta = \ldots$, the result follows by (1). The rule is: Interchange α and β in the first and change sign, which gives the second term; then β and γ in second and change sign; then γ and δ in the third, changing sign.

13017. (A. S. Eve, M.A.)—AB, CD are chords of a circle at right angles; a straight line APQ meets CD in P and the circle in Q. If R is taken in AQ so that AR is a mean proportional between AP and AQ, find (1) the equation of the locus of R, and trace the curve; and (2) solve the same problem, if AR is an arithmetic mean between AP and AQ.

Solution by Rev. J. L. Kitchin, M.A.; *the* Proposer; *and others.*



Let the chords intersect at O, and let

$$\angle PAB = \theta,$$

$$AO = c,$$

and let AB make an angle α with the diameter length ∂. If

$$AR = r,$$

$$r^2 = c \sec\theta \,.\, \partial \cos(\alpha+\theta);$$

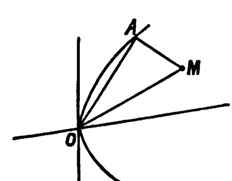
$$\therefore \quad x(x^2+y^2) = c\partial(x\cos\alpha - y\sin\alpha).$$

The roots of r are equal and opposite.

$x = 0$ is an asymptote.

$y = x\tan\alpha$ is the tangent at the origin.

13040. (Professor Cochez.)—Etant donnée une parabole $y^2 = 4ax$, on mène une droite OA et en A une perpendiculaire à cette droite. Puis on construit le triangle rectangle AMO semblable à un triangle donné: (1) lieu de M quand OA pivote autour de O; (2) lieu des foyers de cette courbe.



Solution by H. W. CURJEL, M.A.; Professor CHAKRIVARTI; *and others.*

Since $\angle$ AOM and the ratio OM : OA are constant, the locus of M is a parabola which may be got from the given one by turning it round O through an $\angle$ AOM, and increasing its linear dimensions in the ratio OM : OA. Hence, if the focus of the given parabola is F, the locus of the focus of the locus of M when $\angle$ AOM varies is the line through F at right angles to OF.

13058. (C. E. BICKMORE, M.A.)—Prove that a prime of the form $4m+1$ is always a factor of m^m-1. [The PROPOSER considers this theorem an easy deduction from a well-known property of the "Theory of Numbers," but does not consider the solution complete without a proof of that property.]

Solution by the PROPOSER, Professor NATH COONDOO, *and others.*

If $$\rho^2 = -1, \quad 2\rho = (1+\rho)^2;$$

$$\therefore \quad 2^{2m}\rho^{2m} = (1+\rho)^{4m} \quad \text{and} \quad 2^{2m}\rho^{2m}(1+\rho) = (1+\rho)^{4m+1} \;\ldots\ldots\; (1).$$

Now, if $(1+x)^{4m+1} = c_0+c_1x+c_2x^2+\ldots+c_{4m+1}x^{4m+1}$, $c_0 = c_{4m+1} = 1$, and if $(4m+1)$ be a prime, it is a factor of $c_1, c_2, c_3, \ldots, c_{4m}$; also

$$2^{2m}\rho^{2m} = (-4)^m.$$

Hence (1) becomes

$$(-4)^m+(-4)^m\rho = (1-c_2+c_4-\ldots+c_{4m})+(c_1-c_3+c_5-\ldots+1)\rho \;\ldots\; (2);$$

$\therefore$ (equating possible and impossible parts)

$$(-4)^m = 1-c_2+c_4-\ldots+c_{4m} = c_1-c_3+c_5-\ldots+1 = 1+(4m+1)\,\mathrm{M}\ldots(3).$$

Also $$(-4m)^m = \{1-(4m+1)\}^m = 1+(4m+1)\,\mathrm{N};$$

$$\therefore \quad m^m\{1+(4m+1)\,\mathrm{M}\} = 1+(4m+1)\,\mathrm{N};$$

whence "m^m-1 is a multiple of $4m+1$ if $4m+1$ is a prime number."

(3) is part of the rule for the quadratic character of 2. The proof of it is suggested by CAUCHY's proof in his *Théorie des Nombres*, p. 451.

[Mr. CURJEL gives the following proof:—

Let a be a primitive root of p $(= 4m+1)$.

Then a^m+1 and a^m-1 are prime to p,

and $$a^{2m} \equiv -1 \pmod p, \quad i.e., \quad (a^m+1)^2 \equiv 2a^m,$$

and $$(a^m+1)^{4m} \equiv 2^{2m}(a^{2m})^m \equiv 2^{2m}(-1)^m \equiv (-4)^m,$$

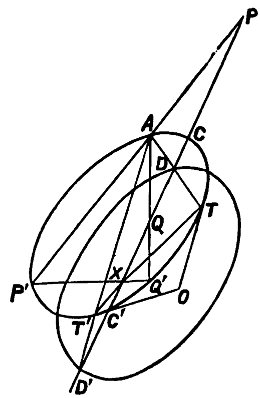
but, since a^m+1 is prime to p, $(a^m+1)^{4m} \equiv 1$; $\therefore (-4)^m \equiv 1$,

but $$(-1)^m \equiv (4m)^m \equiv 4^m \,.\, m^m;$$

$$\therefore \quad 4^m \,.\, m^m \,.\, 4^m \equiv 2^{4m} \,.\, m^m \equiv m^m \equiv 1.$$

13065. (J. E. Campbell, M.A.)—Show how to construct, *with the aid of the ruler only*, a conic passing through a given point, and through the intersections of two conics on each of which five points are known.

Solution by Profs. Gopalachanar, Lampe, *and others.*



Draw any line through P the given point. Construct on it the involution of points conjugate to one of the conics S′. Let A be one of the given points on the other conic S. Project the range from A on to S. The lines joining corresponding points will then pass through a fixed point O. [All of these constructions only involve the use of the ruler, and do not pre-suppose the conics drawn as in the figure.] Find the polar of O with respect to S, and let it intersect the given line in X. Join AP, meeting S again in P′. Join P′X, meeting S again in Q′. Join AQ′, meeting given line in Q. Q is the point where the conic required intersects the line drawn through P.

Let C, C′, D, D′ be the pairs of points where the given line intersects S and S′; and T, T′ the intersections of the polar of O with S. Because AT and AT′ are double lines of the involutions projected on to S, they intersect the given line in D and D′. Now P′Q′, TT′, CC′ form an involution (they pass all through X); therefore PQ, DD′, CC′ form an involution—that is, Q is on the conic of the pencil $S + kS' = 0$, which passes through P.

13036. (Professor Neuberg.)—Etant donnés un tétraèdre ABCD et un point quelconque M, on mène par M des plans parallèles aux quatre faces; ces plans rencontrent les arêtes des trièdres opposés en douze points appartenant à une même quadrique dont on demande l'équation.

Solution by H. W. Curjel, M.A.; Prof. Mukhopadhyay; *and others.*

Let $\alpha_1, \beta_1, \gamma_1, \delta_1$ be the tetrahedral coordinates of M.

Then the planes through M parallel to the α and β planes cut the edge $\gamma\delta$ in $(\alpha_1, 1-\alpha_1, 0, 0)$, $(1-\beta_1, \beta_1, 0, 0)$.

These points lie on the quadric

$$A\alpha^2 + B\beta^2 + C\gamma^2 + D\delta^2 - E\alpha\beta - F\alpha\gamma - G\alpha\delta - H\beta\gamma - J\beta\delta - K\gamma\delta = 0,$$

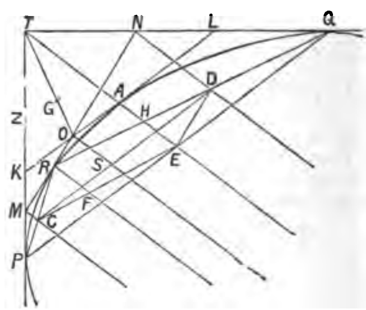
if $$\frac{A}{(1-\alpha_1)/\alpha_1} = \frac{B}{(1-\beta_1)/\beta_1} = \frac{E}{(1-\alpha_1-\beta_1+2\alpha_1\beta_1)/(\alpha_1\beta_1)}.$$

From this result and the symmetrical ones, it is clear that the twelve points lie on the quadric

$$\frac{1-\alpha_1}{\alpha_1}\alpha^2+\frac{1-\beta_1}{\beta_1}\beta^2+\frac{1-\gamma_1}{\gamma_1}\gamma^2+\frac{1-\delta_1}{\delta_1}\delta^2$$
$$-\frac{1-\alpha_1-\beta_1+2\alpha_1\beta_1}{\alpha_1\beta_1}\alpha\beta-\frac{1-\alpha_1-\gamma_1+2\alpha_1\gamma_1}{\alpha_1\gamma_1}\alpha\gamma-\frac{1-\alpha_1-\delta_1+2\alpha_1\delta_1}{\alpha_1\delta_1}\alpha\delta$$
$$-\frac{1-\beta_1-\gamma_1+2\beta_1\gamma_1}{\beta_1\gamma_1}\beta\gamma-\frac{1-\beta_1-\delta_1+2\beta_1\delta_1}{\beta_1\delta_1}\beta\delta-\frac{1-\gamma_1-\delta_1+2\gamma_1\delta_1}{\gamma_1\delta_1}\gamma\delta=0$$

13063. (R. Knowles, B.A.)—Tangents from a fixed point T meet a parabola in P and Q; a variable tangent meets these in M, N, respectively. Prove that the locus of the centroid of the triangle TMN is a right line parallel to PQ.

Solution by W. C. Stanham; G. E. Crawford, M.A.; *and others.*



The theorem will follow if it be shown that O, the point where MN intersects KL, the tangent parallel to PQ, is the middle point of MN.

Let R be the point of contact of MN, and through N, T, O, R, and M draw lines parallel to the axis.

Then RC = CP,

and RD = DQ;

also PE = EQ;

therefore RDEC is a parallelogram, as is also RHEF.

But RA, and therefore RH, is bisected by OS; therefore S is the point of intersection of the diagonals of these two parallelograms;

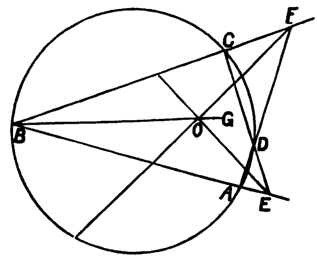
$$\therefore\ \text{CS} = \text{SD},\quad \text{and}\quad \therefore\ \text{MO} = \text{ON}.$$

Therefore, if TZ = 2ZK, the locus of the centroid is the line through Z parallel to PQ.

13064. (R. F. Davis, M.A.)—ABCD is a quadrilateral inscribed in a circle; BA, CD produced meet in E, and AD, BC in F. Prove that the internal bisectors of the angles at E and F (1) are at right angles; (2) meet in a point O, which divides the lines joining the middle points of the diagonals AC, BD in the ratio of the diagonals; (3) form by their

intersection with the sides of the quadrilateral ABCD a rhombus whose side $= AC.BD/(AC+BD)$.

Solution by H. W. Curjel, M.A.; C. E. Crawford, M.A.; *and others.*



Produce BO to G. Then

$$\angle FOE = \angle FOG + \angle GOE$$
$$= \angle OFB + \angle OBF + \angle OBE + \angle OEB$$
$$= \tfrac{1}{2}(2 \text{ right angles} - A - B + 2 \text{ right angles} - C - B) + B$$
$$= \tfrac{1}{2}(4 \text{ rt. angles} - A - C)$$
$$= \text{a right angle.}$$

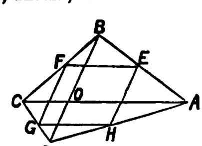
The ratio in which CD, AB are divided by FO $=$ FC : FD $=$ CA : BD
$=$ ratio in which AD, CB are divided by EO;
therefore EO, FO are collinear with the diagonals of an inscribed parallelogram (clearly a rhombus) of the quadrilateral, and therefore intersect at a point which divides the line joining the middle points of the diagonals in the same ratio as that in which the angular points of the rhombus divide the sides of ABCD, *i.e.*, in the ratio AC : BD.

Also, the side of a rhombus : BD $=$ FC : FC + FD $=$ AC : AC + BD;

$\therefore$ side of rhombus $= (BD \,.\, AC)/(AC+BD)$.

13050. (Professor Swaminatha Aiyar.)—In a given quadrilateral a parallelogram is inscribed, whose sides are parallel to the diagonals of the quadrilateral; prove that the diagonals of all such parallelograms intersect on the line which joins the middle points of the diagonals of the quadrilateral, and that the area of the greatest of such parallelograms is half that of the quadrilateral.

Solution by G. Heppel, M.A.; G. E. Crawford, M.A.; *and others.*



OA, OB are axes.

$$OA = a, \quad OB = b, \quad OC = c, \quad OD = d;$$
$$AE = m.AB.$$

Then centre of parallelogram is

$$\left\{\tfrac{1}{2}(1-m)(a-c), \quad \tfrac{1}{2}m(b-d)\right\},$$

and this is a point on the join of the mid-points of diagonals, namely, $(a-c)\,y + (b-d)\,x - \tfrac{1}{2}(a-b)(b-d) = 0.$

$$\text{Area of parallelogram} = m(b+d)(1-m)(a+c)\sin AOB.$$

This is a maximum when $m-m^2$, or $\tfrac{1}{4}-(m-\tfrac{1}{2})^2$ is a maximum; that is, when $m = \tfrac{1}{2}$; and parallelogram is half the quadrilateral.

13055. (Editor.)—If AB, CD be the principal axes of an ellipse, and P the point where the ellipse is cut by a diagonal of the rectangle through A, B, C, D that circumscribes the ellipse, prove that APB, CPD are together equal to two right angles.

Solution by W. C. Stanham; W. E. Heal, M.A.; *and others.*

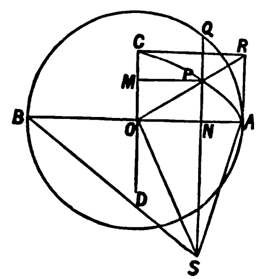
Let OPR be a diagonal. Draw the ordinate QPNS, Q being on the auxiliary circle, and S taken so that

$$NS/NQ = OA/OC.$$

We have $OM/ON = OC/OA$

and $PN/NQ = OC/OA$;

$$\therefore \quad NQ = ON.$$



Therefore the triangle ASB is similar to CPD, its linear dimensions being in a constant ratio (OA/OC) to those of CPD.

And $PN \,.\, NS = \dfrac{OC}{OA} NQ \,.\, \dfrac{OA}{OC} NQ$

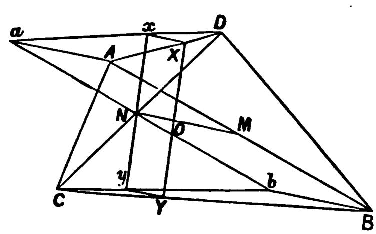
$= NQ^2 = AN \,.\, NB$; $\therefore$ A, P, B, and S are concyclic;

$$\therefore \quad APB + ASB = APB + CPD = \text{two right angles.}$$

13045. (Professor Dupin.)—Tout plan qui passe par les milieux de deux arêtes opposées d'un tétraèdre divise ce solide en deux parties équivalentes.

Solution by H. W. Curjel, M.A.; W. E. Heal, M.A.; *and others.*

Let edges AB, CD of a tetrahedron be bisected in M, N, and let any plane be drawn cutting the edges AD, BC in X, Y. Let MN, XY cut in O. Draw lines through B, A, X, Y parallel to MN, cutting the plane through CD parallel to AB in b, a, x, y. Then, clearly, $\triangle$s bNC, aND are equal in all respects, and



$$Nx = Ny \quad \text{and} \quad Cy : yb = Dx : xa; \qquad \therefore \quad OX = OY.$$

Hence, as X and Y move along DA, CB from D and C to X and Y, the $\triangle$s XMN, YMN describe equal volumes;

$$\therefore \quad \text{volume CYMN} = \text{volume XDMN}.$$

But tetrahedron CDMA = tetrahedron CDMB;

therefore the plane MXNY bisects the tetrahedron ABCD.

3977 & 4004. (Professor CROFTON, F.R.S.)—(3977.) If two points are taken at random within a given circle, find the chance that (1) the inner of the two, (2) the outer, shall lie on a given concentric circle; and (3) extend this problem to cases where more than two points are taken, or to other cases of boundary than a concentric circle.

(4004.) Two points are taken at random within a circle, and the one furthest from the centre is then effaced. Two more are taken in like manner, and the operation repeated an infinite number of times. Determine the law of the distribution of the points which remain. Likewise, if the *nearest* one had been effaced. Extend the question to three or more points.

Solution by H. W. CURJEL, M.A.; Professor SANJÁNA; *and others.*

(4004.) The chance that the first point is at a distance r is $2\pi r dr/\pi R^2$; combining this with the chance R^2-r^2/R^2 that the other is further from the centre, we get that the density at distance r is proportional to R^2-r^2. Or, if n points are taken, density is proportional to $(R^2-r^2)^{n-1}$. In the same way, combining this chance r^2/R^2 that the other point is nearer the centre, density is proportional to r^2 or $r^{2(n-1)}$ when n points are taken.

(3977.) (1) Hence the required chance

$$= \iint (R^2-r^2)^{n-1}\, r\,dr\,d\theta \Big/ \int_0^R \int_0^{2\pi} (R^2-r^2)^{n-1}\, r\,dr\,d\theta$$

(integral being taken over given area)

$$= \int_0^r (R^2-r^2)^{n-1}\, r\,dr \Big/ \int_0^R (R^2-r^2)^{n-1}\, r\,dr$$

(when given area is a concentric circle, radius r)

$$= (2R^2-r^2)\, r^2/R^4 \quad (\text{when } n=2).$$

(2) Required chance

$$= \iint r^{2n-1}\, dr\,d\theta \Big/ \int_0^R \int_0^{2\pi} r^{2n-1}\, dr\,d\theta \quad \text{(integral over given area)}$$

$$= \int_0^r r^{2n-1}\, dr \Big/ \int_0^R r^{2n-1}\, dr \quad \text{(area a concentric circle, radius } r)$$

$$= r^{2n}/R^{2n}.$$

3877. (Professor TAIT, F.R.S.)—Show that, whatever functions of x be represented by y and z, we have always

$$\frac{\int yz\,dx}{\int y\,dx} > \epsilon^{\left(\int y \log z\,dx\right) / \left(\int y\,dx\right)},$$

all the integrals being taken between the same limits of x, and all the quantities involved being positive.

Solution by Professors RAMACHANDRA ROW, CHAKRIVARTI, *and others.*

Consider y_1 quantities equal to z_1, y_2 to z_2, &c.
Then, since the arithmetic mean is greater than the geometric,

$$\frac{y_1 z_1 + y_2 z_2 + \dots}{y_1 + y_2 + \dots} > (z_1^{y_1} z_2^{y_2} \dots)^{1/y_1+y_2+\dots} > e^{(y_1 \log z_1 + y_2 \log z_2 + \dots)/(y_1+y_2+\dots)}.$$

Suppose $y_1, y_2, \dots$, and z_1, z_2 to be functions; we get

$$\frac{\Sigma y_1 z_1}{\Sigma y_1} > e^{(\Sigma y_1 \log z_1)/(\Sigma y_1)}.$$

Proceeding to the limit,

$$\frac{\int yz\,dx}{\int y\,dx} > e^{\left(\int y \log z\,dx\right) / \left(\int y\,dx\right)}.$$

[The above proof seems to assume $y_1 y_2 \dots$ to be positive integers; but this is not necessary for the demonstration. It may be proved that

$$\frac{y_1 z_1 + y_2 z_2}{y_1 + y_2} > (z_1^{y_1} z_2^{y_2})^{1/y_1+y_2},$$

if y_1 and y_2 are positive, though not integers; and the proof may be easily extended in a manner similar to the proof of the inequality between arithmetical and geometrical means.]

12916. (Professor NAGLE.)—Show that the volume included between the surface represented by the equation $z = e^{-(x^2+y^2)}$ and the xy plane equals the square of the area of the section by the zx plane, the limits of x and y being plus and minus infinity.

Solution by Rev. E. S. LONGHURST, B.A.; Professor ZERR; *and others.*

$$\text{Volume indicated} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^z dx\,dy\,dz$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

$$= 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy = 4\int_0^{\infty} e^{-x^2}\,dx \,.\int_0^{\infty} e^{-y^2}\,dz$$

(since x and y are independent)

$$= 4\left\{\int_0^{\infty} e^{-x^2}\,dx\right\}^2.$$

Again, area denoted $= \int_{-\infty}^{+\infty} z\,dx = 2\int_0^{\infty} e^{-x^2}\,dx$ (since $y = 0$ is plane zx).

Hence we obtain the result given.

12769. (R. CHARTRES.)—Sum the infinite series
$\frac{1}{1.3}+\frac{1}{2.5}+\frac{1}{3.7}+$ &c.; and hence deduce $\int_0^{\frac{1}{2}\pi}\sin x \,.\, \log\sin x \,.\, dx$.

Solution by the PROPOSER, Professor SANJÁNA, *and others.*

$$\frac{1}{1.3}+\frac{1}{2.5}+\frac{1}{3.7}\ldots = 2\left\{\tfrac{1}{2}-\tfrac{1}{3}+\tfrac{1}{4}-\tfrac{1}{5}\ldots\right\} = 2(1-\log 2);$$

but $\int_0^{\frac{1}{2}\pi}\sin x \,.\, \log\sin x \,.\, dx = \frac{1}{2}\int_0^{\frac{1}{2}\pi}\sin x\left(-\cos^2 x-\frac{\cos^4 x}{2}-\frac{\cos^6 x}{3}\ldots\right)dx$

$$= -\tfrac{1}{2}\left(\frac{1}{1.3}+\frac{1}{2.5}+\frac{1}{3.7}\ldots\right);$$

$$\therefore \quad = \log 2 - 1.$$

12890. (EDITOR.)—Prove that the locus of the point of concourse of three tangent lines, mutually at right angles, to the paraboloid $y^2/b+z^2/c=4x$ is the paraboloid of revolution $y^2+z^2=4\left\{(b+c)x+bc\right\}$.

Solution by W. C. STANHAM; Professor COONDOO; *and others.*

The equation of the tangent cone from $(\alpha\beta\gamma)$ to $y^2/b+z^2/c=4x$ is

$$(\beta^2/4b+\gamma^2/4c-\alpha)(y^2/4b+z^2/4c-x) = \left\{\beta y/4b+\gamma z/4c-\tfrac{1}{2}(x+\alpha)\right\}^2.$$

The condition that this should have three mutually perpendicular generators is that the sum of the coefficients of x^2, y^2, and z^2 should be zero. This gives $\quad \beta^2+\gamma^2-4(b+c)\alpha-4bc=0.$
Therefore the required locus has for its equation

$$y^2+z^2=4\left\{(b+c)x+bc\right\},$$

a paraboloid of revolution.

9797. (W. J. C. SHARP, M.A.)—If $P\frac{d^2y}{dx^2}+2Q\frac{dy}{dx}+Ry=X$, where P, Q, R, X are functions of x only, and are subject to the condition

$$\frac{d}{dx}\left(\frac{P}{Q}\right)+\frac{PR}{Q^2}-1=0,$$

show that $\quad y=\epsilon^{-\int Q/P\,dx}\iint\frac{X}{P}\,\epsilon^{\int Q/P\,dx}\,dx^2.$

Solution by H. J. WOODALL, A.R.C.S.; Prof. MUKHOPADHYAY; *and others.*

Reduce the given equation to a somewhat simpler form by the substitutions $Q/P=p$, $R/P=q$, $X/P=r$. The given equation, condition,

and solution become $d^2y/dx^2 + 2p\, dy/dx + qy = r$, $dp/dx + p^2 = q$, and

$$y = \exp\left(-\int p\, dx\right) \iint r \exp\left(\int p\, dx\right) dx^2.$$

Multiply the given equation by $\exp\left(\int p\, dx\right)$, and we get

$$\left\{\exp\left(\int p\, dx\right) d^2y/dx^2 + p \exp\left(\int p\, dx\right) dy/dx\right\}$$
$$+ p\left\{\exp\left(\int p\, dx\right) dy/dx + p \exp\left(\int p\, dx\right) y\right.$$
$$+ y \exp\left(\int p\, dx\right) \{q-p^2\} = r \exp\left(\int p\, dx\right).$$

This is $\quad d/dx\left\{\exp\left(\int p\, dx\right) dy/dx\right\} + p\, d/dx\left\{\exp\left(\int p\, dx\right) y\right\}$

$$+ y \exp\left(\int p\, dx\right) dp/dx = r \exp\left(\int p\, dx\right),$$

if $q-p^2 = dp/dx$. Integrate, and we get

$$\exp\left(\int p\, dx\right) dy/dx + p \exp\left(\int p\, dx\right) y = \int r \exp\left(\int p\, dx\right) dx.$$

Integrate again, and we get, finally,

$$y = \exp\left(-\int p\, dx\right) \iint r \exp\left(\int p\, dx\right) dx^2 \ldots \text{ as given.}$$

9579. (Professor WOLSTENHOLME.)—If p be a positive integer, $\alpha, \beta, \gamma, \ldots$ the roots of the equation $x^p = 1$, n any numerical quantity >1, the only real value of $\alpha^{1/n} + \beta^{1/n} + \gamma^{1/n} + \ldots$ is $\tan \dfrac{\pi}{n} \Big/ \tan \dfrac{\pi}{pn}$.

Solution by H. J. WOODALL, A.R.C.S.; Prof. RADHAKRISHNAN; *and others.*

By DEMOIVRE's theorem $\alpha, \beta, \ldots = \cos 2\pi\, l/p + \iota \sin 2\pi\, l/p$; therefore

$$\alpha^{1/n}, \beta^{1/n}, \text{\&c.} = \cos 2\pi\, (l+pm)/pn + \iota \sin 2\pi\, (l+pm)/pn.$$

Sum $= C + \iota S$ say, $= C =$ real if $S = 0$,

$$S = \sin\left[\pi\left\{2m/n + (p-1)/pn\right\}\right] \sin(\pi/n) / \sin(\pi/pn);$$

$$\therefore \quad \sin\left[\pi\left\{2m/n + (p-1)/pn\right\}\right] = 0; \quad \therefore \quad \cos \text{ of this} = \pm 1;$$

$$C = \cos\left[\pi\left\{2m/n + (p-1)/pn\right\}\right] \sin(\pi/n) / \sin(\pi/pn)$$
$$= \pm \sin(\pi/n)/\sin(\pi/pn).$$

9547. (Professor MATZ, M.A.)—Reduce to elliptic forms and integrate the expression $\int (a^4 \pm 2b^2x^2 + x^4)^{\frac{1}{2}}\, dx.$

Solution by H. J. WOODALL, A.R.C.S.; Professor SARKAR; *and others.*

If $b > a$, we may resolve this into two real factors involving x^2,

i.e., $(a^4 \pm 2b^2x^2 + x^4)^{\frac{1}{2}} = \left\{x^2 \pm b^2 \pm (b^4 - a^4)^{\frac{1}{2}}\right\}^{\frac{1}{2}} \left\{x^2 \pm b^2 \mp (b^4 - a^4)^{\frac{1}{2}}\right\}^{\frac{1}{2}}.$

Write $b^2 + (b^4 - a^4)^{\frac{1}{2}} = a_1^2, \quad b^2 - (b^4 - a^4)^{\frac{1}{2}} = a_2^2; \quad \therefore \ a_1 > a_2.$

$$x = a_1 \cot\theta; \qquad \therefore \ x^2 + a_2^2 = \left\{a_1^2 - (a_1^2 - a_2^2)\sin^2\theta\right\}/\sin^2\theta.$$

$$dx = -a_1 \operatorname{cosec}^2\theta \,.\, d\theta.$$

The integral now becomes (using the upper sign)

$$= \int \frac{(x^2 + a_1^2)(x^2 + a_2^2)}{(x^2 + a_1^2)^{\frac{1}{2}}(x^2 + a_2^2)^{\frac{1}{2}}}\, dx = \int \frac{a_1 \operatorname{cosec}\theta \left\{a_1^2 - (a_1^2 - a_2^2)\sin^2\theta\right\}/\sin^2\theta}{\left\{a_1^2 - (a_1^2 - a_2^2)\sin^2\theta\right\}^{\frac{1}{2}} \operatorname{cosec}\theta}$$
$$(-)\, a_1 \operatorname{cosec}^2\theta \,.\, d\theta$$

$$= -a_1 \int \frac{a_1^2 - (a_1^2 - a_2^2)\sin^2\theta}{\sin^4\theta\,(1 - k^2\sin^2\theta)^{\frac{1}{2}}}\, d\theta, \text{ where } k^2 = (a_1^2 - a_2^2)/a_1^2$$

$$= \int \frac{-a_1^3}{\sin^4\theta \,.\, \Delta}\, d\theta + a_1 (a_1^2 - a_2^2) \int \frac{d\theta}{\sin^2\theta \,.\, \Delta} = -a_1^3 V_2 + a_1 (a_1^2 - a_2^2) V_1;$$

using Δ to denote $(1 - k^2\sin^2\theta)$, and V_2, V_1 is the usual way.

To find connexion between V_n and V_0, we have

$$\frac{d}{d\theta}\left\{\frac{\sin\theta\cos\theta\,(1 - k^2\sin^2\theta)^{\frac{1}{2}}}{(\sin^2\theta)^{n-1}}\right\}$$
$$= -\frac{(2n-3)}{(\sin^2\theta)^{n-1} \,.\, \Delta} + \frac{(1+k^2)(2n-4)}{(\sin^2\theta)^{n-2} \,.\, \Delta} - \frac{(2n-5)k^2}{(\sin^2\theta)^{n-3} \,.\, \Delta},$$

integrate, and we get

$$\sin\theta\cos\theta \,.\, \Delta/(\sin^2\theta)^{n-1}$$
$$= -(2n-3)V_{n-1} + (2n-4)(1+k^2)V_{n-2} - (2n-5)k^2 V_{n-3}.$$

If $n = 3$, we find

$$3V_2 = 2(1+k^2)V_1 - k^2 V_0 - \sin\theta\cos\theta\,(1 - k^2\sin^2\theta)^{\frac{1}{2}}/\sin^4\theta.$$

Here $\quad V_1 = \Pi(k, \lambda, \theta), \quad V_0 = F(k, \theta) \quad$ (but $\lambda = \infty$).

Making this substitution, we get

$$\text{former integral} = -a_1^3 V_2 + a_1 (a_1^2 - a_2^2) V_1$$

$$= -(\tfrac{1}{3}a_1^3 + \tfrac{1}{3}a_1 a_2^2)V_1 + \tfrac{1}{3}a_1 (a_1^2 - a_2^2)V_0 + \tfrac{1}{3}a_1^3 \cos\theta\,(1 - k^2\sin^2\theta)^{\frac{1}{2}}/\sin^3\theta$$
$$= \tfrac{1}{3}a_1^3(F - \Pi) - \tfrac{1}{3}a_1 a_2^2 (F + \Pi) + \tfrac{1}{3}a_1^3 \cos\theta\,(1 - k^2\sin^2\theta)^{\frac{1}{2}}/\sin^3\theta.$$

If the lower signs be taken, we must substitute $x = a_1 \operatorname{cosec}\theta$. The rest of the work will be similar to that above.

12854. (Professor MATZ.)—If A walk to the City and ride back, he will require $m = 5\frac{1}{4}$ hours; but, if he walk both ways, he will require $n = 7$ hours. How many hours will he require to ride both ways?

Solution by Rev. S. J. ROWTON, M.A., Mus.D.; R. CHARTRES; *and others.*

Assuming the rates to be uniform, he walks there in $3\frac{1}{2}$ hours; therefore he rides back in $5\frac{1}{4} - 3\frac{1}{2} = 1\frac{3}{4}$ hours; therefore he can ride both ways in $3\frac{1}{2}$ hours.

12779. (Professsor DE VOLSON WOOD.)—A prismatic bar, having a uniform angular velocity W and a linear velocity of v feet per second, suddenly snaps (by the disappearance of the cohesive force) into an indefinite number of equal parts; required the resultant angular velocity of each piece and the locus of the parts at the end of t seconds after rupture.

Solution by H. W. CURJEL, M.A.; Professor CHAKRIVARTI; *and others.*

The velocity of each particle will be that compounded of v and $\mathrm{W}r$ at right angles to the direction of r, where r is the distance of the particle from the axis of rotation through the centre of gravity of the bar at the time of rupture; the angular velocity of each part will clearly be W; and the locus of the parts will be a distorted image of the bar, still prismatic, but lengthened in all directions at right angles to the axis of rotation in the ratio $\qquad (1+\mathrm{W}^2t^2)^{\frac{1}{2}} : 1.$

5988. (Professor MATZ, M.A.)—A right cone, of altitude a (= 24 ins.) and radius of base m (= 7 ins.), is pierced by an auger of radius n (= 3 ins.), the centre of the auger passing perpendicularly through the axis of the cone at a distance of b (= 6 ins.) from the centre of the base. Find the volume removed.

Solution by H. J. WOODALL, A.R.C.S.; Prof. COONDOO; *and others.*

Take axes x along axis of cone, y along axis of hole, and z perpendicular to both.

Radius of circle (section of cone) at height $x+b$ from base

$$= m(a-x-b)/a.$$

At this height $y = \{m^2(a-x-b)^2 - a^2z^2\}^{\frac{1}{2}}/a$,

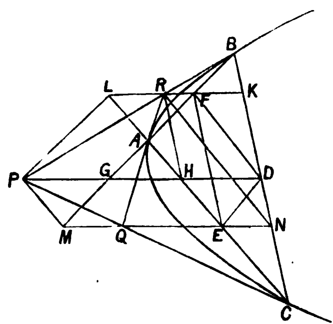
$$\text{volume removed} = \frac{4}{a}\int_{-n}^{n}\int_0^{(n^2-x^2)^{\frac{1}{2}}} \{m^2(a-x-b)^2 - a^2z^2\}^{\frac{1}{2}}\,dz\,.\,dx$$

$$= \frac{2}{a}\int_{-n}^{n}\Big[m^2(a-x-b)^2 \text{arc sin}\,\{a(n^2-x^2)^{\frac{1}{2}}/m(a-x-b)\}$$

$$+ (n^2-x^2)^{\frac{1}{2}}\{m^2(a-x-b)^2 - a^2(n^2-x^2)\}^{\frac{1}{2}}\Big]\,dx.$$

13089. (R. F. DAVIS, M.A.)—A series of parabolas are described through three given points. Prove that the tangents at these points to any one of the curves form a triangle whose angular points lie respectively on three fixed hyperbolas having two of the sides of the triangle formed by the fixed points as asymptotes and the third side as tangent.

Solution by H. W. CURJEL, M.A.; W. C. STANHAM; *and others.*



Let A, B, C be the given points, and let PQR be the triangle formed by the tangents at A, B, C to any parabola through A, B, C. Let D, E, F be the middle points of BC, CA, AB. Let PD meet BA, CA in G and H; RF meet BC, CA in K, L; QE meet AB, BC in M, N.

Now RF, PD, QE are parallel to the axis of the parabola, and the parabola clearly bisects GH and PD; hence we have

$$PH = GD = ME.$$

Therefore PM is parallel to AC.

Similarly PL is parallel to AB;

$\therefore$ parallelogram PMAL = PMEH = AEDF.

Therefore P lies on the hyperbola touching BC at D, and having AB, AC as asymptotes; *i.e.*, locus of P is that hyperbola.

Similarly, since parallelogram RHCN = FECD and parallelogram QKBG = BFED, the loci of R and Q are, respectively, the hyperbola touching BA at F, with asymptotes BC, CA, and the hyperbola touching AC at E, having asymptotes BA, BC.

13113. (J. J. WALKER, F.R.S.) — Show that the perpendicular vector on the line of intersection of the planes through the terms of the vectors $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ is

$$(V\Sigma\beta\gamma\, S\alpha'\beta'\gamma' - V\Sigma\beta'\gamma'\, S\alpha\beta\gamma)\, V^{-1}.\, V\Sigma\beta'\gamma'\, V\Sigma\beta\gamma.$$

Solution by Rev. J. CULLEN, Professor MUKHOPADHYAY, *and others.*

Let $\rho = \delta + x\sigma$ be the equation of the line of intersection, where δ is the required perpendicular vector and σ a unit vector in the intersection of the planes. For the sake of brevity, write $V\Sigma\beta\gamma = \theta$, $V\Sigma\beta'\gamma' = \theta'$, and $\psi = \theta S\alpha'\beta'\gamma' - \theta' S\alpha\beta\gamma$.

Then $S\rho\theta - S\alpha\beta\gamma = 0$, $S\rho\theta' - S\alpha'\beta'\gamma' = 0$ (TAIT, *Q.*, § 209); $\therefore S\rho\psi = 0$;

also, $$\sigma = UV\theta'\theta.$$

Hence ψ is perpendicular to the plane containing ρ, δ, and σ;

$$\therefore \quad \delta = y V \psi \sigma.$$

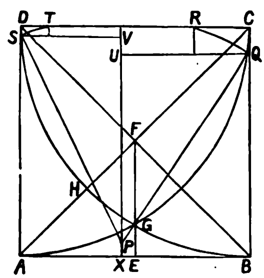
Operating with $S.\theta$ on $\rho = yV\psi\sigma + x\sigma$, we find $y = -1/TV\theta'\theta$;

$$\therefore \quad \delta = -V\psi\sigma . T^{-1} V\theta'\theta = V\psi\sigma^{-1} . T^{-1} V\theta'\theta,$$

since $\sigma^2 = -1$; therefore, &c.

13080. (Professor FINKEL.)—Prove that the chance that the distance of two points within a square shall not exceed a side of the square is $\frac{39}{40}$.

Solution by D. BIDDLE, Professor RADHAKRISHNAN, *and others.*



Let ABCD be the square. By the law of symmetry, it will suffice to consider the first point as placed in AEF, which is one of eight similar triangles composing the square. AEF may be divided into three portions, namely, AEG, which allows of corner-spaces at both C and D for the second point, beyond the specified distance; AHG, which allows of such a corner-space at C only; and FGH, which allows of no such corner-space. BEG has the same relation to corner-spaces at C that AEG has to those at D; and AHD is in respect to C like AHB. Consequently we can integrate for the whole space ABHDA, and multiply by 4 instead of 8. Let P be the first point, x, y its coordinates taken along CD, CB, respectively, and PQ = PR = AB. Then CQR is the corner-space referred to, and its value is

$$xy - \tfrac{1}{2}\left\{x(1-x^2)^{\frac{1}{2}} + y(1-y^2)^{\frac{1}{2}} + \sin^{-1}x - \cos^{-1}y\right\}.$$

Consequently the required probability

$$= 1 - 4\int_0^1 \int_{(1-x^2)^{\frac{1}{2}}}^1 \left[xy - \tfrac{1}{2}\left\{x(1-x^2)^{\frac{1}{2}} + \sin^{-1}x + y(1-y^2)^{\frac{1}{2}} - \cos^{-1}y\right\}\right] dx\,dy$$

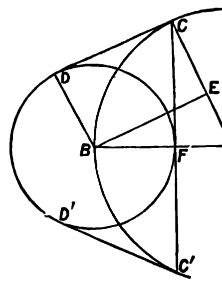
$$= 1 - 4\int_0^1 \left\{x - \tfrac{1}{6}x^3 - \tfrac{1}{2}x(1-x^2)^{\frac{1}{2}} - \tfrac{1}{2}\sin^{-1}x\right\} dx$$

$$= 1 - 4\left\{\tfrac{1}{2} - \tfrac{1}{24} - \tfrac{1}{2}(1-\tfrac{2}{3}) - \tfrac{1}{2}(\tfrac{1}{2}\pi - 1)\right\} = 1 - (3\tfrac{1}{6} - \pi) = \cdot 9749259,$$

or $\frac{39}{40}$, nearly.

13083. (Professor GOPALACHANAR.)—A circle A passes through the centre of a circle B; prove that their common tangents will touch A in points lying on a tangent to B.

Solution by W. E. Heal; H. W. Curjel, M.A.; *and others.*



Let the common tangents be CD, C'D'.

Let CC' cut AB in F.

Draw BE perpendicular to AC.

Then CF cuts AB at right angles, and triangles ACF, ABE are equal in all respects;

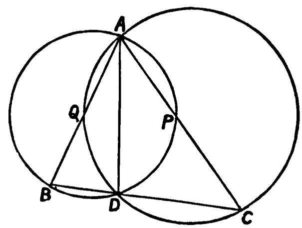
$\therefore \quad AE = AF;$

$\therefore \quad BF = CE = BD;$

$\therefore$ CFC' touches circle B.

13084. (Editor.)—If AD be a line drawn from the vertex A to the side BC of a triangle ABC, and the circum-circles of ABD, ACD cut AC, AB in P, Q, investigate (1) the relation between the segments BQ, CP; and find what this becomes when AD bisects (2) the angle A, (3) the triangle ABC.

Solution by H. W. Curjel, M.A.; Professor Coondoo; *and others.*



$$CP . CA = CB . CD,$$

$$BA . BQ = BC . BD;$$

$$\therefore \quad CP . CA + BQ . BA = BC^2.$$

If AD bisects $\angle$ A, we have

$$\frac{CP . CA}{BA . BQ} = \frac{CB . CD}{BC . BD} = \frac{C}{BA};$$

$$\therefore \quad CP = BQ.$$

If AD bisects BC,

$$CP . CA = BA . BQ = \tfrac{1}{2} BC^2.$$

13115. (D. Biddle.)—On the circumference of a circle is a fixed point A; there are also $2n+1$ other points taken at random, so that each single point is anywhere on the circumference, and the position, in relation to A, of the middle one of the $2n+1$ points is registered. If an infinite number of such sets of $2n+1$ points be taken, prove (1) that the density of the middle ones varies as $x^n(1-x)^n$, where x is the length of arc measured from A, the entire circumference being regarded as unity; (2) that the density of the middle points is therefore greatest opposite A. If the positions of the 1st, 2nd, 3rd, ..., $(2n+1)$th points be similarly registered, those in each set being reckoned in order from A, after their random disposition, prove (3) that their *relative* densities, at any part of

the circumference distant x from A, will be given by the successive terms in the expansion of $\{(1-x)+x\}^{2n}$; and (4) that their respective *average* densities are identical.

Solution by H. W. CURJEL, M.A.; Professor FINKEL; *and others.*

The chance that any point is in the arc ABC (length x) is x, and that it is in arc ADC (length $1-x$) is $1-x$. Hence the chance that r points are in arc ABC, and $2n-r$ in ADC, is ${}^{2n}C_r x^r (1-x)^{2n-r}$; therefore the density of the middle point varies as ${}^{2n}C_n x^n (1-x)^n$, *i.e.*, as $x^n(1-x)^n$; this is clearly greatest when $x = 1-x$, that is, at the point opposite A. Also, the relative densities of the first, second, third, &c. points are proportional to the successive terms of $\{(1-x)+x\}^{2n}$.

Their respective average densities are evidently equal, for there are as many rth points as there are sth points; this may be seen as follows:—

$$\text{average density} = \int_0^1 {}^{2n}C_r\,(1-x)^r x^{2n-r}\,dx = \frac{1}{2n+1} = \text{constant}.$$

13117. (R. LACHLAN, Sc.D.)—Prove that the periodic continued fraction
$$\frac{1}{a_1+}\,\frac{1}{a_2+}\cdots\frac{1}{a_k+}\,\frac{1}{a_1+}\cdots = \frac{p_k}{q_k\pm}\,\frac{1}{x\pm}\,\frac{1}{x\pm}\,\frac{1}{x\pm}\cdots,$$
where $x = p_{k-1}+q_k$, and the upper or lower sign is to be taken according as k is odd or even. Show that the nth convergent of the latter form is equal to the nkth convergent of the former.

Solution by C. E. BICKMORE, M.A.; Professor LAMPE; *and others.*

Calling, with Professors CAYLEY and SYLVESTER, the numerator of the continued fraction $a+\dfrac{1}{b+}\,\dfrac{1}{c+}\ldots$, the cumulant $(abc\ldots)$,

$$p_k = (a_2a_3\ldots a_{k-1}a_k),\quad p_{k-1} = (a_2a_3\ldots a_{k-1}),\quad q_k = (a_1a_2a_3\ldots a_{k-1}a_k),$$
$$q_{k-1} = (a_1a_2a_3\ldots a_{k-1}),\quad \text{and}\quad (p_kq_{k-1}-p_{k-1}q_k) = (-1)^k,$$
$$p_{2k} = (a_2a_3\ldots a_{k-1}a_ka_1a_2a_3\ldots a_k) = p_kq_k+p_{k-1}p_k = p_kx,$$
$$q_{2k} = (a_1a_2a_3\ldots a_{k-1}a_ka_1a_2a_3\ldots a_k) = q_kq_k+q_{k-1}p_k = q_kx-(-1)^k;$$

hence, as k be odd or even, $p_{2k}/q_{2k} = p_k/(q_k+1/x)$, $p_{2k}/q_{2k} = p_k/(q_k-1/x)$; similarly, $p_{3k} = xp_{2k}+(-1)^{k-1}p_k$, $q_{3k} = xq_{2k}+(-1)^{k-1}q_k$, &c.; whence, by so-called mathematical induction,

$$p_{nk} = xp_{(n-1)k}+(-1)^{k-1}p_{(n-2)k},\quad q_{nk} = xq_{(n-1)k}+(-1)^{k-1}q_{(n-2)k}.$$

Now if P_n/Q_n be the nth convergent to the continued fraction on the right hand,

$$P_n = xP_{n-1}+(-1)^{k-1}P_{n-2},\quad Q_n = xQ_{n-1}+(-1)^{k-1}Q_{n-2},$$
$$\therefore\quad P_n = p_{nk},\quad Q_n = q_{nk}.$$

The function $x = (a_1a_2a_3\ldots a_{k-1}a_k)+(a_2a_3\ldots a_{k-1})$ is unaltered by

reversing the order of the constituents, or by beginning the cycle with any constituent other than a_1.

[Mr. Curjel gives the solution in this way:—

Let P_n/Q_n be the nth convergent of the latter continued fraction. Then

$$p_k/q_k = P_1/Q_1,$$

$$\frac{p_{2k}}{q_{2k}} = \frac{(q_k/p_k)\,p_k+p_{k-1}}{(q_k/p_k)\,q_k+q_{k-1}} = \frac{q_kp_k+p_kp_{k-1}}{q_k^2+q_kp_{k-1}+p_kq_{k-1}-q_kp_{k-1}} = \frac{xp_k}{xq_k\pm 1} = \frac{P_2}{Q_2};$$

$\therefore$ $P_n/Q_n = p_{nk}/q_{nk}$, when $n=1$ and when $n=2$. Suppose it is true for all values of n up to n; then it may easily be shown that

$$P_n = Q_{n-1}p_k+p_{k-1}P_{n-1} \quad \text{and} \quad Q_n = Q_{n-1}q_k+P_{n-1}q_{k-1}$$

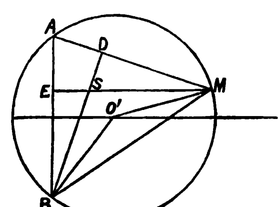
for all values of n up to n.

Then
$$\frac{p_{(n+1)k}}{q_{(n+1)k}} = \frac{(q_{nk}/p_{nk})\,p_k+p_{k-1}}{(q_{nk}/p_{nk})\,q_k+q_{k-1}} = \frac{Q_np_k+P_np_{k-1}}{Q_nq_k+P_nq_{k-1}}$$
$$= \frac{p_k\,(xQ_{n-1}\pm Q_{n-2})+p_{k-1}\,(xP_{n-1}\pm P_{n-2})}{q\;\;(xQ_{n-1}\pm Q_{n-2})+q_{k-1}\,(xP_{n-1}\pm P_{n-2})}$$
$$= \frac{x\,(P_{n-1}p_{k-1}+Q_{n-1}p_k)\pm(p_kQ_{n-2}+p_{k-1}P_{n-2})}{x\,(Q_{n-1}q_k+P_{n-1}Q_{k-1})\pm(q_kQ_{n-2}+q_{k-1}P_{n-2})}$$
$$= \frac{xP_n\pm P_{n-1}}{xQ_n\pm Q_{n-1}} = \frac{P_{n+1}}{Q_{n+1}};$$

therefore, by induction, $P_n/Q_n = p_{nk}/q_{nk}$ universally.

13081. (Professor Morel.)—Etant données deux circonférences O et O′, qui se coupent aux points A et B, on joint un point quelconque M de la circonférence O′ aux points A, B, et on prolonge ces deux droites, s'il y a lieu, jusqu'à leur rencontre en P et Q avec la circonférence O. Trouver, relativement au triangle MPQ, le lieu géométrique (1) du point de concours des hauteurs; (2) des pieds des hauteurs issues des sommets P et Q; (3) du pied de la hauteur issue du sommet M. (Ce dernier lieu n'est pas une conique.)

Solution by H. W. Curjel, M.A.; Professor Krishmanacharry; *and others.*



Take O′ as origin and axis of y parallel to AB.

Let equation to circle MAB be

$$x^2+y^2 = a^2,$$

and equation to AB

$$y+a\cos\alpha = 0.$$

Now

PQ = a constant × AB = k . AB,

and MA . MP = MB . MQ.

Therefore, if S is orthocentre of

$\triangle$MAB, R, the orthocentre of $\triangle$MPQ is found by producing MO′ to R, so that
$$MR = k \, . \, MS.$$

(1) Now $\quad MS = 2a \cos \alpha; \qquad \therefore \quad O'R = a - 2ak \cos \alpha;$

$\therefore$ locus of R is the circle $x^2 + y^2 = a^2 (1 - 2k \cos \alpha)^2$.

(2) Draw BD perpendicular to AM; let $\angle O'MS = \theta$; then
$$\angle O'MB = \tfrac{1}{2}(\alpha - \theta), \quad \angle SMB = \tfrac{1}{2}(\alpha + \theta),$$
and
$$MD = MB \cos \alpha = 2a \cos \tfrac{1}{2}(\alpha - \theta) \cos \alpha.$$

Therefore foot of perpendicular from P on MBQ is given by
$$x = a \cos \theta - 2ka \cos \tfrac{1}{2}(\alpha - \theta) \cos \alpha \cos \tfrac{1}{2}(\alpha + \theta)$$
$$= a \cos \theta (1 - k \cos \alpha) - ka \cos^2 \alpha,$$
$$y = a \sin \theta - 2ka \cos \tfrac{1}{2}(\alpha - \theta) \cos \alpha \sin \tfrac{1}{2}(\alpha + \theta)$$
$$= a \sin \theta (1 - k \cos \alpha) - ka \cos \alpha \sin \alpha;$$

$\therefore$ locus is the circle
$$(x + ka \cos^2 \alpha)^2 + (y + ka \cos \alpha \sin \alpha)^2 = a^2 (1 - k \cos \alpha)^2.$$

Similarly, the locus of the foot of the perpendicular from Q on MAP is the circle $(x + ka \cos^2 \alpha)^2 + (y - ka \cos \alpha \sin \alpha)^2 = a^2 (1 - k \cos \alpha)^2$.

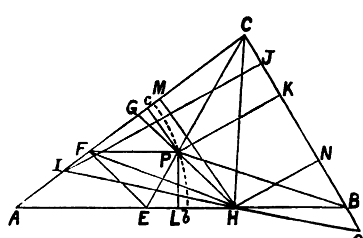
(3) Let MS cut AB in E; then $ME = a (\cos \alpha + \cos \theta)$.

Therefore foot of perpendicular from M on PQ is given by
$$x = a \cos \theta (1 - ka \cos \alpha - ka \cos \theta), \qquad y = a \sin \theta (1 - ka \cos \alpha - ka \cos \theta).$$

Therefore locus is $\quad (x^2 + y^2 + xka)^2 = (1 - k \cos \alpha)^2 a^2 (x^2 + y^2)$.

13023. (I. Arnold.)—Find a point at a given distance from the vertex of a given triangle so that the sum of the three perpendiculars therefrom on the sides of the triangle shall be equal to a given right line; and determine the limits.

Solution by M. Brierley, Professor Swaminatha Aiyar, *and others.*



Let ABC be the given triangle, and IJ, drawn perpendicular to BC, the base, and meeting AC in I, equal to the given right line. With centre A and radius equal to the given distance from the vertex, describe an arc of circle cb, meeting AC, AB in c and b, then take BE, CF (each = BC) in AB, AC, respectively, and join EF. Also, produce CB to O, so that CO = CI, and join F, O, cutting AB in H; draw HG parallel to EF meeting the arc cb in P; P is the point required.

Demit the perpendiculars HN, PK upon BC; HM, PG upon AC; PL upon AB; and join A, P; B, P; C, P; P, F; H, F; C, H; and P, E. The quadrilateral $CFEB = CPF + CPB + EPB + EPF$

$$= CHF + CHB + FHE;$$

but $EPF = EHF$; $\therefore$ $CPF + CPB + EPB = CHF + CHB$.

Because ICO is isosceles, $HM + HN = IJ$;

and, since $BE = CF = BC$, $PQ + PK + PL = HM + HN = IJ$.

Obviously, HG is the locus of P, and, as the distance from the vertex is greater and less than AG, AH, respectively, so is the problem limited to the interior of the triangle.

13066. (Rev. T. C. Simmons, M.A.)—*Problem.*—"Three points being taken at random on the circumference of a circle, what is the probability that they all lie on the same semi-circle?" *Solution.*—"Let A, B, C be the points. Then A, B must both lie on some one semi-circle terminated at A. Chance that C lies on this same semi-circle $= \frac{1}{2}$. Therefore, chance that A, B, C all lie on the same semi-circle $= \frac{1}{2}$." But the correct answer ought to be $\frac{3}{4}$, and the above solution contains a fallacy. Point it out.

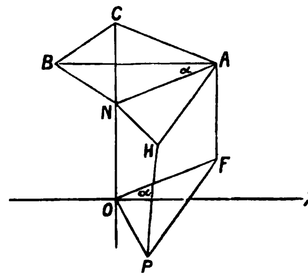
Solution by H. W. Curjel, M.A.; Rev. J. L. Kitchin, M.A.; *and others.*

The fallacy consists in not taking account of the fact that C may be on the same semi-circle as A and B, and at the same time be on the opposite side of A to B. The proof should be as follows:—"Take any positions of A, B. Chance that B and C are on opposite sides of diameter through A is $\frac{1}{2}$; chance that A and C are on opposite sides of diameter through B is $\frac{1}{2}$; therefore chance that A, B, C are *not* on the same semi-circle is $\frac{1}{4}$; therefore chance that they are on the same semi-circle is $\frac{3}{4}$."

[The Proposer is of opinion that this new solution, as it stands, is unsatisfactory. The probability of a compound event is not formed from the product of the probabilities of the component events unless the latter are *independent*. Now it is by no means easy to see that the positions of B and C relatively to the diameter through A are independent of the positions of A and C relatively to the diameter through B.]

12977. (Professor Chakrivarti.)—An engineer besieging a town receives information that the powder magazine lies at a given distance S.E. from the bottom of a flag-staff, the *top* C of which is visible above the wall of the town from a rising ground at some distance from the town. On this eminence, the altitude of which above the level of the town is known, he erects a battery A; he then measures the horizontal base AB in a direction due west, and from its extremities observes the angles of elevation of C as well as the angles CAB, CBA. Show that from these data the distance and bearing of the magazine from the battery may be found.

Solution by W. C. STANHAM, Professor SWAMINATHA, *and others.*



Let O be the foot of the flagstaff, PH vertical, POF a horizontal plane, AF being vertical, and P the powder magazine, ONC the flagstaff, and B, N, H points in the horizontal plane through A; OX a line from west to east.

Then $\text{FOX} = \text{BAN} = \alpha$.

From the angles CAB, CBA and the length AB, CA and BC are found.

From CAN and CA, AN is found.

From CBN and CB, BN is found.

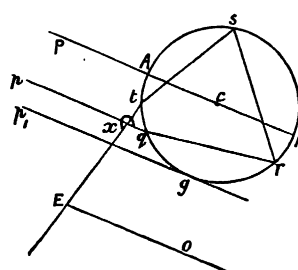
From AN, BN, and AB, α is found, and therefore $\text{ANH} = 45^\circ + \alpha$ is known. And $\text{NH} = \text{OP}$ is given. From NH, AN, and ANH, AH and the angle NAH are found. $\text{PH} = \text{AF}$ is also given. Therefore AP and the angle HAP can be found. And $\text{BAH} = \alpha + \text{NAH}$ is known.

12548. (Professor SANJÁNA, M.A.)—"Show that, if β be the angular radius of the secondary bow corresponding to any value of μ,

$$\sin \tfrac{1}{2}\beta = \left[(\mu^2+1)(9-\mu^2)^3\right]^{\frac{1}{2}}/8\mu^3."$$

Prove that this result (given in ALDIS's *Optics*) is wrong, and find the correct value of $\sin \frac{1}{2}\beta$.

Solution by Professors SANJÁNA, KRISMACHANDRA DE, *and others.*



Suppose a pencil of parallel rays incident on a spherical rain-drop, and emergent after two internal reflections.

Let the rays be parallel to the diameter AcA', and let *pqrste* be the course of that ray of the system which passes with minimum deviation.

With the usual notation,

$$\begin{aligned} D &= (\phi-\phi') + (\pi-2\phi') + (\pi-2\phi') \\ &\qquad + (\phi-\phi') \\ &= 2\pi + 2\phi - 6\phi': \end{aligned}$$

$$\therefore\quad dD/d\phi = 2\left\{1 - 3\cos\phi/(\mu\cos\phi')\right\}.$$

Hence, for this ray,

$\mu\cos\phi' = 3\cos\phi$; also, $\mu\sin\phi' = \sin\phi$; thus $\mu^2 = 9\cos^2\phi + \sin^2\phi$.

This gives $\cos^2\phi = (\mu^2-1)/8$, $\sin^2\phi = (9-\mu^2)/8$,

$\cos^2\phi' = (9\mu^2-9)/8\mu^2$, $\sin^2\phi' = (9-\mu^2)/8\mu^2$.

It is easily seen that rays incident on the arc Aq (not extremely near to pq) emerge from the arc $A'rq$ in a state of divergence; while those incident on the arc qg emerge from the arc At, and form by their intersection a caustic of which Et is an asymptote.

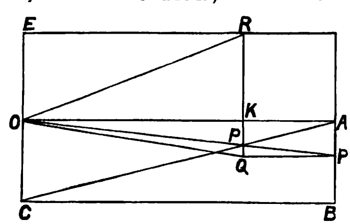
Hence, to an eye conveniently situated at E, there will be the impression of illumination in the direction Et. If EO be parallel to AA′, all drops whose centres lie on a cone of axis EO and semi-vertical angle OEt will produce a similar impression. Hence the angle OEt is the radius β of the bow.

Now $\quad OEt = Exp = D - \pi; \quad \therefore \quad \beta = \pi + 2\phi - 6\phi'.$

Thus
$$\sin \tfrac{1}{2}\beta = \cos(3\phi' - \phi)$$
$$= \cos\phi' \cos\phi\,(4\cos^2\phi' - 3) + \sin\phi' \sin\phi\,(3 - 4\sin^2\phi')$$
$$= \frac{1}{16\mu^3}(2\mu^4 + 36\mu^2 - 54) = (\mu^4 + 18\mu^2 - 27)/8\mu^3.$$

13011. (Professor Davidoglou.)—Par le sommet O d'un rectangle OABC, on mène une droite variable Δ qui coupe la diagonale AC en P et le côté AB en P_1. Prouver que le lieu du point Q, intersection des parallèles à AB et BC, menées respectivement par P et P_1, est une hyperbole.

Solution by H. W. Curjel, M.A.; Professor Sarkar; *and others.*



Produce CO to E, making

$$OE = CO.$$

Draw ER parallel to OA. Produce PQ to meet OA and ER in K and R.

$\Delta ORQ = \Delta OKC + \Delta OKQ.$

But $QK : OA = P'A : OA$
$= PK : OK;$

$\therefore \quad \Delta QKO = \Delta OPA;$

$$\therefore \quad \Delta ORQ = \Delta OAC = \text{constant};$$

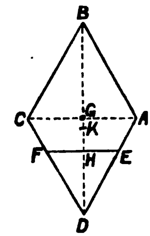
therefore locus of Q is a rectangular hyperbola passing through A with asymptotes EC, ER.

13037. (Professor Nath Coondoo.)—Four equal and similar rods are loosely jointed at their extremities, and the frame so formed is suspended freely from one of the angular points; it is prevented from closing by a smooth rod resting symmetrically on the two lower rods, this rod being of the same material and thickness, and of a length equal to one nth of that of each of the four rods. Prove that the angle which the sides of the frame make with the vertical is given by the equation

$$\operatorname{cosec}^3\theta = 4n(2n+1), \quad \text{provided that } 2n \text{ is greater than } \sqrt{3}+1.$$

Solution by G. Heppel, M.A.; Professor Mukhopadhyay; *and others.*

G is centre of gravity of the four rods, H of EF, and K of the system.



The length of AB being taken as unity,

$$BK = \{4n(2n+1)\cos\theta - \cot\theta\} / 2n(4n+1).$$

Differentiating, we find this a maximum, when

$$\operatorname{cosec}^3\theta = 4n(2n+1).$$

The condition that EF should rest on the lower rods is that $1/2n$ should be less than $\sin\theta$; so that $2n^2-2n-1$ must be positive, and $2n$ must be greater than $\sqrt{3}+1$.

If $2n$ has any less value, EF coincides with AC, and the system of reactions has an altered character.

12984. (J. J. Walker, F.R.S.)—If the perimeter of a spherical triangle ABC is a quadrant, show that the difference between the cosine and sine of any side is equal to the product of the tangents of the halves of the adjacent angles.

Solution by H. W. Curjel, M.A.; Professor A. Droz-Farny; *and others.*

$$\tan\tfrac12 B\tan\tfrac12 C = \left\{\frac{\sin(s-a)\sin(s-c)}{\sin s\,.\,\sin(s-b)}\,.\,\frac{\sin(s-a)\sin(s-b)}{\sin s\,.\,\sin(s-c)}\right\}^{\frac12}$$

$$= \pm\frac{\sin(s-a)}{\sin s}\quad\text{(the sign being taken which makes this fraction positive)}$$

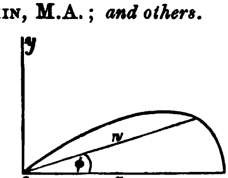
$$= \sin a - \cos a;\ \text{since } s = \tfrac14\pi.$$

13034. (J. W. West.)—A solid is generated by the rotation of Bernouilli's lemniscata about the axis of (y); find its volume and surface.

Solution by the Proposer; Rev. J. L. Kitchin, M.A.; *and others.*

Polar equation $\phi = a(\cos 2\phi)^{\frac12}$.

V = volume. S = surface.



$$S = 4\pi a^2\int_0^{\frac14\pi}\cos\phi\,d\phi = 2\sqrt{2}\pi a^2.$$

$$V = \frac{4\pi a^3}{3}\int_0^{\frac14\pi}(\cos 2\phi)^{\frac32}\cos\phi\,d\phi$$

$$= \frac{\pi a^3}{6\sqrt{2}}\left\{(\cos 2\phi)^{\frac12}(2\cos 2\phi+3)(1-\cos 2\phi)^{\frac12} - 3\sin^{-1}(\cos 2\phi)^{\frac12}\right\}_0$$

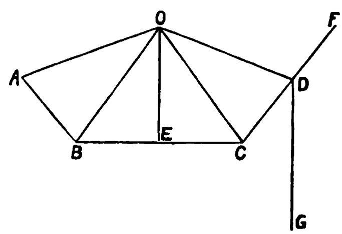
$$= \frac{\pi^2 a^3}{4\sqrt{2}} = \tfrac14\pi,\ \text{volume circumscribing cylinder.}$$

12964. (W. J. Dobbs, M.A.)—OABCD is a framework of rods smoothly jointed at their extremities, the rods OA, OB, OC, OD being each of length 25 inches; the rods AB, CD each of length 14 inches; and the rod BC of length 30 inches. Two bodies weighing 100 lbs. each are suspended from A and D respectively, and the whole is supported at O. The rods themselves being of no appreciable weight, find the stress in each rod.

Solution by H. W. Curjel, M.A.; Professor Chakrivarti; *and others.*

$\angle \mathrm{OCD} = \sin^{-1}\frac{24}{25} = 2\sin^{-1}\frac{3}{5} = \angle \mathrm{BOC}.$

Produce CD to F, and let DG be drawn vertically downwards through D. Let T, Q, P, R be the tensions along OD, OC and the pressures along CD, BC.



Then

$$\frac{100 \text{ lbs.}}{\sin \mathrm{ODF}} = \frac{\mathrm{T}}{\sin \mathrm{FDG}} = \frac{\mathrm{P}}{\sin \mathrm{ODG}}.$$

$\therefore \quad \mathrm{T} = \frac{125}{2}$ lbs.,

and $\quad \mathrm{P} = \frac{195}{2}$ lbs.

Again
$$\frac{\mathrm{Q}}{\sin \mathrm{ECD}} = \frac{\mathrm{P}}{\sin \mathrm{OCE}} = \frac{\mathrm{R}}{\sin \mathrm{OCD}},$$

$$\mathrm{Q} = \mathrm{P} = \tfrac{195}{2} \text{ lbs.} \quad \text{and} \quad \mathrm{R} = \tfrac{195}{2} \times \tfrac{5}{6} \text{ lbs.} = \tfrac{325}{4} \text{ lbs.}$$

12337. (Professor Ch. Hermite, LL.D.)—Prouver la formule

$$\int_0^\pi \frac{\sin x\, dx}{\sin(x-a)} = e^{cia}\pi.$$

On doit supposer, dans cette formule, $a = \alpha + i\beta$, β étant essentiellement différent de zéro, et prendre pour c la valeur $+1$ ou -1, suivant que β est positif ou négatif.

Solution by Professors Sanjána, Mukhopadhyay, *and others.*

As $\qquad \sin x/\sin(x-a) = \cos a + \sin a \cot(x-a),$

the given integral $= \pi \cos a + \sin a \int_0^\pi \cot(x-a)\, dx$

$$= \pi\cos a + \sin a \left[\log \sin(x-a)\right]_0^\pi = \pi \cos a + \sin a \log(-1)$$

$$= \pi \cos a \pm \pi \,.\, i \sin a \quad \text{(taking principal values only)} = \pi e^{cia}.$$

12847. (Professor RAMASWAMI AIYAR, M.A. Suggested by Quest. 12245, which Professor AIYAR believes to be incorrect.)—If A and B be positive integers, such that x^2+Ax+B and x^2+Ax-B are both resolvable into simple factors, show that A and B can be expressed, and in one way only, in the forms $A = \lambda(m^2+n^2)$, $B = \lambda^2 mn(m^2-n^2)$, where λ, m, n are integers and m, n are mutual primes, one of which is even.

Solution by H. W. CURJEL, M.A.; Professor BHATTACHARYA; *and others.*

For simple factors,

$$A^2+4B = y^2, \quad A^2-4B = z^2, \quad 8B = y^2-z^2, \quad 2A^2 = y^2+z^2;$$

$$\therefore \quad m(y-A) = (A-z)n, \quad n(y+A) = m(A+z),$$

where m and n are prime to one another;

$$\therefore \quad \frac{A}{m^2+n^2} = \frac{y}{m^2+2mn-n^2} = \frac{z}{n^2+2mn-m^2} = \lambda;$$

$$\therefore \quad A = \lambda(m^2+n^2), \quad B = \lambda^2 mn(m^2-n^2).$$

If m, n are both odd, we may write $(m+n+1)$, $(m-n)$ for m, n, and we get $\quad A = 2\lambda\left\{(m+n+1)^2+(m-n)^2\right\}$,

$$B = (2\lambda)^2(m-n)(m+n+1)\left\{(m+n+1)^2-(m-n)^2\right\},$$

which are of the forms in the Question, after taking out any common factor of $m-n$, $m+n+1$, and multiplying λ by its square.

If there are two values of m, n, one odd and one even, then it can easily be shown that they are of the form

$$(ax-by), \ (ay+bx); \quad (ax+by), \ (ay-bx);$$

and $\quad (ax-by)(ay+bx)(ax-by+ay+bx)(by-ax+ay+bx)$

is then unaltered by changing the sign of b;

i.e., $\quad \left\{xy(a^2-b^2)+ab(x^2-y^2)\right\}\left\{(a^2-b^2)(y^2-x^2)+4abxy\right\}$

is unaltered by changing the sign of b;

$$\therefore \quad -ab(x^2-y^2)^2(a^2-b^2)+4abx^2y^2(a^2-b^2) = 0;$$

$$\therefore \quad (x^2-y^2)^2 = 4x^2y^2; \qquad i.e., \quad (y\pm x)^2 = 2x^2,$$

which is impossible in whole numbers.

Hence A, B can be expressed in the forms given in the Question, and in one way only.

13069. (R. W. D. CHRISTIE.)—Prove (1) the incorrectness or correctness of the following statement from the *Encyclopædia Britannica*:—"Since a sum of three squares into a sum of three squares is not a sum of three squares," Vol. xv., Art. "Number." (2) Show indirectly that there is, in general, *a test for prime numbers* by *casting out the nines.*

Solution by H. W. CURJEL, M.A.; Professor RADHAKRISHNAN; *and others.*

It is not true that the sum of three squares multiplied by the sum of three squares *cannot* be a sum of three squares; *e.g.*,

$$(a^2+b^2+c^2)^2 = (a^2+b^2-c^2)+4a^2c^2+4b^2c^2,$$

a particular case of the more general theorem that any positive integral power of the sum of n squares is the sum of n squares; also, in many other cases, the product is the sum of three squares; *e.g.*,

$$(2^2+3^2+7^2)(4^2+5^2+6^2) = 69^2+3^2+2^2.$$

12955. (J. J. Barniville, B.A.)—Prove that

$$\frac{7}{1^2 \,.\, 4^2}+\frac{115}{7^2 \,.\, 10^2}+\frac{367}{13^2 \,.\, 16^2}+\frac{763}{19^2 \,.\, 22^2}+\ldots = \frac{4\pi^2}{81}.$$

Solution by Professor Sanjana; H. W. Curjel, M.A.; *and others.*

The general term of the series may be written

$$\tfrac{1}{3}\left\{\frac{1}{(3n-1)^2}+\frac{1}{(6n-2)^2}+\frac{1}{(6n-5)^2}\right\};$$

$$\therefore \quad \text{sum to infinity} = \tfrac{1}{3}\left\{\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots\right\}$$

$$= \tfrac{1}{3}\times\frac{\pi^2}{6}\times(1-\tfrac{1}{9}) = \frac{4\pi^2}{81}.$$

12976. (Professor Nath Coondoo.)—Some merchants form a capital of £8240, to which each contributes forty times as many pounds as there are merchants. With this whole sum they gain as many pounds per cent. as there are merchants. They then divide the profit, and each takes ten times as many pounds as there are merchants, after which there remains £224 over. How many merchants were there?

Solution by H. W. Curjel, M.A.; Rev. J. L. Kitchin, M.A.; *and others.*

Let x = number of merchants.

Then $$x^3-25x^2+206x-560 = 0;$$

$$\therefore \quad x = 7,\ 8,\ \text{or } 10.$$

9685. (Professor Finkel.)—Find all the roots of the equation

$$x^9-x^8-13x^7+13x^6+33x^5-33x^4-59x^3+59x^2+108x-108 = 0.$$

Solution by H. J. Woodall; Prof. Nath Coondoo; *and others.*

The roots are $\quad x = 1,\ \pm 1{\cdot}59225,\ \pm 3{\cdot}19988,$

and $\quad \pm 1{\cdot}04014 \pm 0{\cdot}98446i.$

12821. (Professor GAUMONT.)—Si dans le polynome $Ax^2+Bxy+Cy^2$ on fait $x = ax_1+by_1$, $y = bx_1-ay_1$, on obtient un nouveau polynome de la forme $A_1x_1^2+B_1x_1y_1+C_1y_1^2$. Démontrer que l'on a

$$B_1^2-4A_1C_1 = (a^2+b^2)^2(B^2-4AC).$$

Solution by Professors A. DROZ-FARNY, SANJANA, *and others.*

Le problème proposé est un cas particulier du théorème général que le discriminant de la transformée d'une forme binaire est égal à celui de la forme primitive multiplié par le carré du module de la transformation. Or le module de la transformation linéaire

$$x = ax_1+by_1, \quad y = bx_1-ay_1,$$

n'est rien d'autre que le déterminant

$$\begin{vmatrix} a & b \\ b & -a \end{vmatrix} = -(a^2+b^2).$$

12851. (Professor SHIELDS.)—Four brothers, A, B, C, D, owned a *round* island, and agreed to divide it by each walking the circumference of a circle or concentric circle. The distance that A walked around the circumference of the centre circle, together with $\frac{1}{3}$ of the distance that B, C, D walked, is equal to 6850 rods; the distance that B walked around the second larger circle, together with $\frac{1}{4}$ of the distance that A, C, D walked, is equal to 6850 rods; the distance that C walked around the third larger circle, together with $\frac{1}{5}$ of the distance that A, B, D walked, is equal to 6850 rods; and the distance that D walked around the circumference of the *island*, together with $\frac{1}{6}$ of the distance that A, B, C walked, is equal to 6850 rods. Find how far each party walked; and, knowing that A got the centre circle and B, C, D got the concentric rings, respectively, how many acres each got.

Solution by T. SAVAGE; Rev. J. L. KITCHIN, M.A.; *and others.*

$$A+\tfrac{1}{3}(B+C+D) = 6850, \qquad B+\tfrac{1}{4}(C+D+A) = 6850,$$
$$C+\tfrac{1}{5}(D+A+B) = 6850, \qquad D+\tfrac{1}{6}(A+B+C) = 6850,$$

Hence $2A+(A+B+C+D) = 3.6850$, $3B+(A+B+C+D) = 4.6850$.

Therefore $3B-2A = 6850$, $4C-A = 2.6850$, $5D-2A = 3.6850$.

Substituting the values of B, C, D in the first of the original equations, we have $A = 2350$ rods, $B = 3850$, $C = 4600$, $D = 5050$.

Also

$$\text{A's area} = \frac{2350\times 2350}{4\pi\times 160}\text{ acres}, \qquad \text{B's} = \frac{(3850+2350)(3850-2350)}{4\pi\times 160},$$

$$\text{C's} = \frac{(4600+3850)(4600-3850)}{4\pi\times 160}, \quad \text{D's} = \frac{9650\times 450}{4\pi\times 160}.$$

13062. (J. Brill, M.A.)—A particle moves under the influence of a conservative field of force, and is subject to a resistance which is proportional to its velocity ($\kappa \times$ velocity). Prove that there exists a function A, such that

$$u = e^{-\kappa t}\frac{\partial A}{\partial x},\quad v = e^{-\kappa t}\frac{\partial A}{\partial y},\quad w = e^{-\kappa t}\frac{\partial A}{\partial z},\quad \tfrac{1}{2}(u^2+v^2+w^2)+Q+e^{-\kappa t}\frac{\partial A}{\partial t}=0,$$

where u, v, w are the components of the velocity of the particle, and Q is the potential of the field of force.

Solution by the Proposer, W. C. Stanham, *and others.*

The equations of motion will be of the type

$$\frac{du}{dt}+\kappa u+\frac{\partial Q}{\partial x}=0,$$

which may be written in the form

$$\frac{d}{dt}(ue^{\kappa t})+e^{\kappa t}\frac{\partial Q}{\partial x}=0.$$

Thus, if we write $U = ue^{\kappa t}$, $V = ve^{\kappa t}$, $W = we^{\kappa t}$,

$$F = \tfrac{1}{2}(U^2+V^2+W^2)\,e^{-\kappa t}+Qe^{\kappa t} = \left\{\tfrac{1}{2}(u^2+v^2+w^2)+Q\right\}e^{\kappa t};$$

Then our equations of motion may be written in the form

$$\frac{dU}{dt}+\frac{\partial F}{\partial x}=0,\quad \frac{dV}{dt}+\frac{\partial F}{\partial y}=0,\quad \frac{dW}{dt}+\frac{\partial F}{\partial z}=0.$$

In addition to these, we readily obtain

$$\frac{dx}{dt}=\frac{\partial F}{\partial U},\quad \frac{dy}{dt}=\frac{\partial F}{\partial V},\quad \frac{dz}{dt}=\frac{\partial F}{\partial W}.$$

These six equations are exactly similar in form to Hamilton's equations for the case of three dependent variables, the only difference being that F is now supposed to contain the time explicitly. Hamilton's results have, however, been extended to this case by Jacobi (*Vorlesungen über Dynamik*) and Donkin (*Phil. Trans.*, 1854). Thus there will exist a function A such that

$$U=\frac{\partial A}{\partial x},\quad V=\frac{\partial A}{\partial y},\quad W=\frac{\partial A}{\partial z},\quad F+\frac{\partial A}{\partial t}=0.$$

From these the results given in the question immediately follow.

10280. (R. W. D. Christie.)—Prove that

$$_nC_1(1^m)-{}_nC_2(2^m)+{}_nC_3(3^m)-\pm{}_nC_n(n^m)\equiv 0.$$

Solution by H. J. Woodall, Professor Gopalachanar, *and others.*

This series is merely the reverse order to that in which is usually obtained the coefficients of x^m on both sides from the identity

$$(e^x-1)^m = (x+x^2/2!+x^3/3!+\ldots)^m.$$

It is evident that it only holds when m is $< n$.

12957. (CECIL EWING.)—Find x, y, z from

$$x^2+yz = a, \quad y^2+xz = b, \quad z^2+xy = c.$$

Solution by Professors A. DROZ-FARNY, NATH COONDOO, *and others.*

En posant, d'après BARDEY's *Gleichungen höheren Grades*, $s = x^2+y^2+x^2$, les équations peuvent s'écrire:

$$(x+y-z)(x-y+z) = 2a-s, \quad (x+y-z)(-x+y+z) = 2b-s,$$
$$(x-y+z)(-x+y+z) = 2c-s.$$

Par multiplication directe

$$(x+y-z)(x-y+z)(-x+y+z) = \{(2a-s)(2b-s)(2c-s)\}^{\frac{1}{2}} = \mathrm{R}.$$

On en déduit

$$x+y-z = (2a-s)(2b-s)/\mathrm{R}, \quad x-y+z = (2a-s)(2c-s)/\mathrm{R},$$
$$-x+y+z = (2b-s)(2c-s)/\mathrm{R}.$$

Par addition

$$x+y+z = [(2a-s)(2b-s)+(2a-s)(2c-s)+(2b-s)(2c-s)]/\mathrm{R};$$

d'où $$x = [(2a-s)(2b-s)+(2a-s)(2c-s)]/2\mathrm{R}$$

et des valeurs analogues pour y et z.

Il suffira de porter ces valeurs dans la relation $s = x^2+y^2+z^2$ d'où une équation du quatrième degré pour s.

12999. (J. M. STOOPS, B.A.)—Find a rational function of $\sin\theta$ and $\cos\theta$ such that $\sin\theta$ and $\cos\theta$ may each be expressed rationally in that function.

Solution by Professor RADHAKRISHNAN, W. C. STANHAM, *and others.*

Let x denote such a function, and let $\sin\theta = f(x)$. Then

$$\cos\theta = \{1-[f(x)]^2\}^{\frac{1}{2}},$$

and these are to be rational functions of x.

A possible form of $f(x)$ is $\{2(ab)^{\frac{1}{2}}/(ax+b/x)\}$,

and $\cos\theta$ is then $\pm\{(ax-b/x)/(ax+b/x)\}$.

And $$x = (b/a)^{\frac{1}{2}}.(1\pm\cos\theta)/\sin\theta.$$

Similarly the formula $x + (b/a)^{\frac{1}{2}}.(1\pm\sin\theta)/\cos\theta$ is obtained.

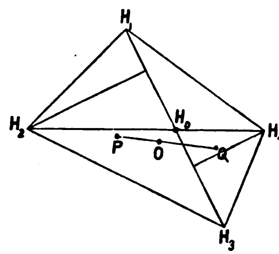
In both of these formulæ, $(b/a)^{\frac{1}{2}}$ may be any factor, independent of $\sin\theta$ and $\cos\theta$, which is itself rational.

Also if c be a rational constant, it is obvious that $x+c$ satisfies the conditions.

4356. (J. J. WALKER, F.R.S.)—If h_1, h_2, h_3, h_4, h_0 are the depths of the four corners and intersection of diagonals, respectively, of any plane quadrilateral below the surface of water, prove that the depth of its centre of pressure is equal to $(\Sigma h_1^2 + \Sigma h_1 h_2 - h_0 \Sigma h_1) / 2(\Sigma h_1 - h_0)$.

Solution by Professors SANJÁNA, KRISHMACHANDRA DE, *and others*.

Let H_1, H_2, H_3, H_4, H_0 be the four corners and diagonal point, respectively; P, Q, O, the centres of pressure on $H_1H_2H_3$, $H_1H_4H_3$, and the quadrilateral, respectively; and p, q, x their depths below the surface; then



$$2p(h_1+h_2+h_3) = h_1^2+h_2^2+h_3^2 + h_1h_2+h_2h_3+h_3h_1,$$

$$2q(h_1+h_4+h_3) = h_1^2+h_4^2+h_3^2 + h_1h_4+h_4h_3+h_3h_1$$

(*Reprint*, XXI., p. 90; BESANT, § 183);

$x =$ (pressure at P $\times p$ + pressure at Q $\times q$) / (sum of pressures at P, Q). Hence

$$x = \frac{\frac{1}{3}g\rho\,(\triangle H_1H_2H_3)(h_1+h_2+h_3)\,p + \frac{1}{3}g\rho\,(\triangle H_1H_4H_3)(h_1+h_4+h_3)\,q}{\frac{1}{3}g\rho\left\{(\triangle H_1H_2H_3)(h_1+h_2+h_3)+(\triangle H_1H_4H_3)(h_1+h_4+h_3)\right\}}$$

$$= \tfrac{1}{2}\cdot\frac{(h_2-h_0)(h_1^2+h_2^2+h_3^2+h_1h_2+h_2h_3+h_3h_1) + (h_0-h_4)(h_1^2+h_4^2+h_3^2+h_1h_4+h_4h_3+h_3h_1)}{(h_2-h_0)(h_1+h_2+h_3)+(h_0-h_4)(h_1+h_4+h_3)}$$

$$= \tfrac{1}{2}\cdot\frac{(h_2-h_4)\left\{h_1^2+h_2^2+h_3^2+h_4^2+\Sigma h_1h_2 - h_0(h_1+h_2+h_3+h_4)\right\}}{(h_2-h_4)(h_1+h_2+h_3+h_4-h_0)}$$

$$= \tfrac{1}{2}\cdot\frac{\Sigma h_1^2+\Sigma h_1h_2-h_0\Sigma h_1}{(\Sigma h_1-h_0)} = \tfrac{1}{2}(\Sigma h_1 - \Sigma h_1h_2/3h),$$

if h is depth of C. of G. of area ABCD.

[These expressions for p (and q) were first given by Mr. WALKER in Question 4276 (1874).]

13057. (VINCENT J. BOUTON, B.Sc., F.R.A.S.)—A circle of radius c moves so that its plane remains parallel to the plane of zx, while its centre describes the circle $x^2+y^2 = a^2$ in the plane of xy. Prove that the equation of the surface generated is $(x^2-y^2+z^2+a^2-c^2)^2 = 4x^2(a^2-y^2)$, and draw figures giving the shape of the surface.

Solution by Rev. J. CULLEN, S.J.; Rev. J. L. KITCHIN, M.A.; *and others*.

Let α and β be the coordinates of the centre of the generating circle;

$$\therefore\quad \alpha^2+\beta^2 = a^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Since its plane is parallel to the plane xz, we have $\beta = y$;

$$\therefore \quad \alpha^2 + y^2 = a^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

The equation of the generating circle is $(x-\alpha)^2 + z^2 = c^2$ .... (3).

Eliminating α between (2) and (3), we obtain

$$(x^2 - y^2 + z^2 + a^2 - c^2)^2 = 4x^2(a^2 - y^2),$$

the required equation of the surface.

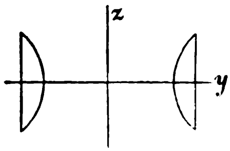


Fig. 1.

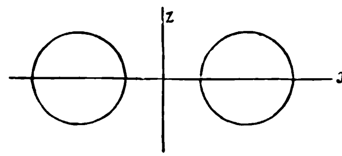


Fig. 2.

Let $a > c$; then, (1) when $x = 0$, $y^2 - z^2 = a^2 - c^2$, the equation of a rectangular hyperbola, so that the intersection of the surface and the plane yz is as in Fig. 1.

(2) When $y = 0$, $(x \pm a)^2 + z^2 = c^2$; therefore the intersection with the plane xz is two circles as in Fig. 2.

(3) When $z = 0$, $(x^2 - y^2 + a^2 - c^2)^2 = 4x^2(a^2 - y^2)$;

$\therefore \quad c^2 = a^2 - y^2 \pm 2x(a^2 - y^2)^{\frac{1}{2}} + x^2$;

$\therefore \quad \pm c = \pm (a^2 - y^2)^{\frac{1}{2}} - x$,

or $\quad (x \pm c)^2 + y^2 = a^2$.

Hence the intersection with the plane xy is represented by Fig. 3, the curves being the circles given by the last equation.

The change in the figures is obvious when $a =$ or $< c$.

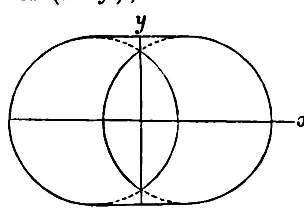


Fig. 3.

[In (3) the intersection with the plane xy is the pair of circles $(x \pm c)^2 + y^2 = a^2$. Even if the area of the moving circle in its extreme positions is supposed to form part of the surface generated, the intersection will consist of the convex figure included by the above pair of circles and their common tangents.]

13085. (Rev. T. C. Simmons, M.A. Suggested by Quest. 12898.)—A, B, C are three particular grains in a stone of rice, which is divided into 14 one-pound parcels, and then dispersed. The chance of separation of A and B, or B and C, or C and A, is now in each case $\frac{13}{14}$. The three events are absolutely independent; the relative positions of A and B, for instance, being in no wise affected by the position of C. Therefore (1) the chance of concurrence of any two of them (for instance, the separation of

A from B, also B from C) is $\frac{13}{14}\cdot\frac{13}{14}$; and (2) the chance of concurrence of all three—*i.e.*, the separation of A from B, also B from C, also C from A—is $\frac{13}{14}\cdot\frac{13}{14}\cdot\frac{13}{14}$. Required, *to point out the fallacy in the above argument;* for, while the first result $\frac{13}{14}\cdot\frac{13}{14}$ is correct, a very simple mode of solution proves the second result ought to be $\frac{13}{14}\cdot\frac{12}{14}$.

Solution by H. W. CURJEL, M.A.; Prof. COONDOO; *and others.*

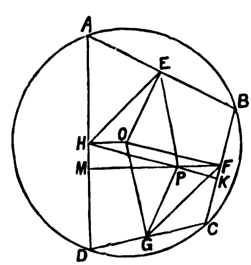
In the favourable case, where A and B are separated and B and C are separated, A and C are excluded from the parcel in which B is. Therefore chance they are separated $= \frac{12}{13}$ (instead of $\frac{13}{14}$, which is obtained by neglecting this fact), which gives the chance of separation of A, B, C

$$= \frac{13}{14}\times\frac{13}{14}\times\frac{12}{13} = \frac{13}{14}\cdot\frac{12}{14}.$$

[The PROPOSER remarks that this is not an answer to the question. He also states that the whole question, with reference to the discussion of Question 12898 also, has been dealt with by him in a paper recently read by him at the Tunis Congress, *Sur les Probabilités des Évènements Composés*, in which he boldly called in question the legitimacy of multiplying, in certain cases even of independent events, the component probabilities together.]

13127. (H. D. DRURY, M.A.)—Prove that the perpendiculars drawn from the middle points of the sides of a quadrilateral inscribed in a circle on the opposite sides are concurrent.

Solution by R. F. DAVIS, M.A.; H. W. CURJEL, M.A.; *and others.*



Let E, F, G, H be the middle points of the sides AB, BC, CD, DA of a quadrilateral inscribed in a circle with centre O. Draw HK, FM perpendicular to BC, AD, and cutting in P. Then OFPH is a parallelogram;

∴ HO is equal and parallel to PF, and EH, FG are both parallel to BD, and equal to $\frac{1}{2}$BD;

∴ EO, PG are equal and parallel;

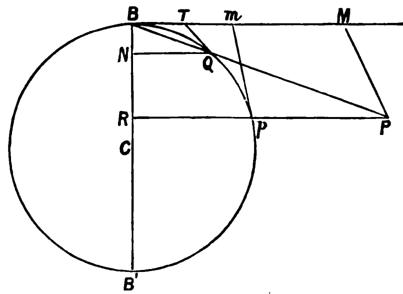
∴ EOGP is a parallelogram;

∴ EP, GP are perpendicular to CD, AB.

13109. (Professor DEZ.)—Etant donnée une ellipse, on mène la tangente à l'extrémité B du petit axe, puis, d'un point M pris sur cette tangente, on mène une seconde tangente MP à la courbe. Trouver le lieu de la projection du point M sur la corde de contact BP.

Solution by H. W. CURJEL, M.A.; Professor CHAKRIVARTI; *and others.*

On the minor axis BB′ as diameter, describe a circle BQpB′ cutting BP in Q. Draw PpR and QN perpendicular to BB′. Let tangents to the circle at p and Q cut BM in m and T. Then we have



$$\text{BT} : \text{B}m = \frac{\text{BN}}{\text{QN}} : \frac{\text{BR}}{\text{R}p}$$
$$= \text{R}p : \text{RP}$$
$$= b : a$$
$$= \text{B}m : \text{BM};$$

$$\therefore \quad \text{BT} : \text{BM} = b^2 : a^2.$$

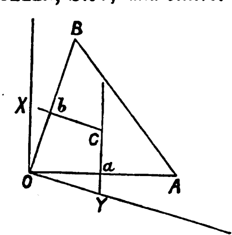
But the perpendicular from T on BQ passes through centre C of ellipse;

$\therefore$ perpendicular from M on BP cuts BB′ in a fixed point K, such that
$$\text{BK} : \text{BC} = a^2 : b^2;$$

$\therefore$ locus is a circle on BK as diameter.

13112. (EDITOR.)—A moveable straight line slides between two fixed straight lines which pass through a given point, and a circle is drawn about the triangle thus formed. Find the envelope of this circle, and the locus of its centre, supposing that the moveable line is (1) of constant length, or (2) cuts off from the fixed lines a triangle of constant area.

Solution by H. W. CURJEL, M.A.; Rev. J. CULLEN, S.J.; *and others.*

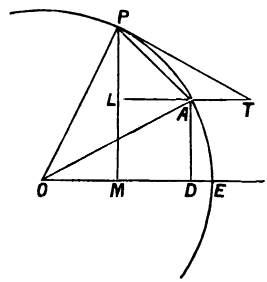


(1) If the moveable line is constant, the radius of the circum-circle is constant ($= k$, say); therefore the envelope of the circle and the locus of its centre are circles with centre the given point and radii $2k$ and k, respectively.

(2) Let OAB be one of the triangles when the area is constant. Draw OX, OY perpendicular to OA, OB; and from centre C of circle OAB draw CaY, CbX perpendicular to OA, OB, meeting them in a, b. Then Oa.Ob = constant, and $\triangle$s ObX, OaY are of constant species; therefore OX.OY is constant; therefore locus of C is a hyperbola with asymptotes OX, OY and the locus of the end of the diameter through O of the circle is a hyperbola with same asymptotes but double its linear dimensions. Hence the envelope of the circle is the pedal of this hyperbola with respect to O, and is, therefore, a bicircular quartic.

13025. (J. M. Stoops, B.A.)—Prove that there is a value of θ between α and x such that $(\sin x - \sin \alpha)/(x-\alpha) = \cos \theta$.

Solution by R. F. Davis, M.A.; Rev. J. L. Kitchin, M.A.; *and others.*



Let O be the centre of a circle of which the radii OP, OA make with the radius OE the angles x, α, respectively $(90° > x > \alpha)$. Draw PM, AD perpendiculars upon OE, and AL upon PM. Let the tangent at P meet LA produced in T. Then

$$\sin x - \sin \alpha = \mathrm{PL}/\mathrm{OP};$$
$$x - \alpha = \text{arc } \mathrm{AP}/\mathrm{OP}.$$

Therefore

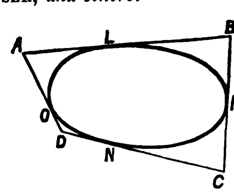
$$(\sin x - \sin \alpha)/(x-\alpha) = \mathrm{PL}/\text{arc } \mathrm{AP},$$

which obviously lies between PL/PT and PL/AP, that is, between $\cos x$ and $\cos \frac{1}{2}(x+\alpha)$.

The given equation has therefore a root between $\frac{1}{2}(x+\alpha)$ and x; *a fortiori* between α and x.

13093. (Rev. T. W. Robinson.) — Give a statical proof of this well-known theorem: "The locus of the centres of four-tangent conics is the straight line bisecting the diagonals."

Solution by I. Arnold, *the* Proposer, *and others.*



Let ABCD be quadrilateral; LONM one of the conics. Take four forces AB, AD, CB, and CD. The resultant will evidently pass through the middle points of BD and of AC, and therefore acts along line through mid-point of diagonals. Again, resultant of AL and AO, which bisects LO, must pass through centre of conic. Similarly of OD and ND, of CN and CM, and of MB and LB. Therefore resultant of original four forces passes through centre of conic, *i.e.*, line through mid-point of diagonals of quadrilateral passes through centre of conic.

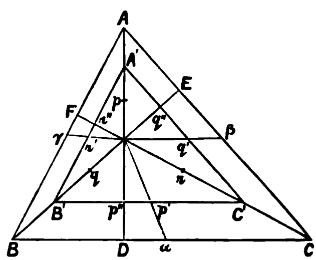
13110. (Professor Morel.) — *Généralisation du cercle des 9 points:* Soient α, β, γ les milieux des côtés BC, CA, AB, d'un triangle; P le point de rencontre des hauteurs AD, BE, CF; O le centre du cercle

circonscrit au triangle dont le rayon est R. Sur les segments PA, PB, PC, Pα, Pβ, Pγ, on prend les points p, q, r, p', q', r' de telle sorte que

$$Pp = 1/n\,PA, \quad Pq = 1/n\,PB, \quad Pr = 1/n\,PC;$$
$$Pp' = 2/n\,P\alpha, \quad Pq' = 2/n\,P\beta, \quad Pr' = 2/n\,P\gamma;$$

et enfin on désigne par p'', q'', r'' les pieds des perpendiculaires abaissées des points p', q', r' sur les hauteurs AD, BE, CF respectivement. Démontrer que p, q, r, p', q', r', p'', q'', r'' sont neuf points d'une même circonférence, dont le rayon est égal à $1/n$ R et dont le centre est un point M situé sur la ligne PO de telle sorte que PM = $1/n$ PO.

Solution by H. W. Curjel, M.A.; Prof. Mukhopadhyay; *and others.*



Produce $p'p''$, $q'q''$, $r'r''$ to meet in A′, B′, C′. Then A′, B′, C′ are clearly on PA, PB, PC; and P is orthocentre of $\triangle$A′B′C′ and p', q', r' the middle points of the sides and p, q, r the middle points of PA′, PB′, PC′. Therefore the 9-point circle of $\triangle$A′B′C′ passes through p, q, r, p', q', r', p'', q'', r''; also the linear dimensions of $\triangle$A′B′C′ are $2/n$ of those of $\triangle$ABC;

$\therefore$ radius of circle M

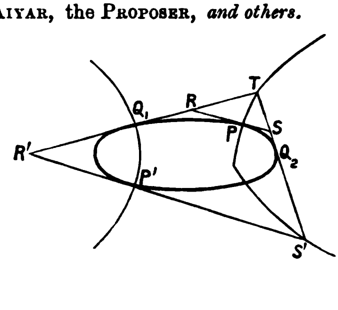
$= 2/n$ of 9-point circle of ABC $= R/n$.

And centre of 9-point circle of ABC bisects PO;

$$\therefore \quad PM = 2/n \times \tfrac{1}{2}PO = PO/n.$$

13097. (W. C. Stanham.) — P and T are any two points on a hyperbola, in the same quadrant of the axes. The tangents from T to the confocal ellipse through P intersect the tangent at P to the ellipse in R and S. Show that TR − TS = PR − PS.

Solution by Prof. Ramaswami Aiyar, the Proposer, *and others.*



This relation, and the relation TR + PR = TS + PS, when T and P are in diagonally opposite quadrants, result from the following considerations:—

(1) The eccentric angle is the same for all points on the hyperbola in the same quadrant.

(2) A confocal ellipse will therefore pass through R and S.

(3) GRAVES's and MACCULLAGH's theorems for elliptic arcs.

For, let α_1, α_2, and ϕ be the eccentric angles of Q_1, Q_2, and P. Then the coordinates of R are

$$a\cos\tfrac{1}{2}(\phi+\alpha_1)/\cos\tfrac{1}{2}(\phi-\alpha_1),\quad b\sin\tfrac{1}{2}(\phi+\alpha_1)/\cos\tfrac{1}{2}(\phi-\alpha_1),$$

with similar expressions for S and T.

If R and S lie on the confocal $x^2/(a^2+\lambda)+y^2/(b^2+\lambda) = 1$, the elimination of λ gives, on reduction,

$$x'^2\sec^2\phi - y'^2\operatorname{cosec}^2\phi = a^2-b^2 = c^2,$$

T being (x', y'), with the condition that, for a possible ellipse (λ positive),

$$(ay'\operatorname{cosec}\phi - bx'\sec\phi) \text{ and } (ax'\sec\phi - by'\operatorname{cosec}\phi + c^2)$$

are of the same sign.

And (1) (which is easily proved) shows that these conditions are satisfied if T and P are in the same, or in diagonally opposite, quadrants.

Then, by GRAVES's theorem,

$$PR+RQ_1-\text{arc } PQ_1 = PS+SQ_2-\text{arc } PQ_2.$$

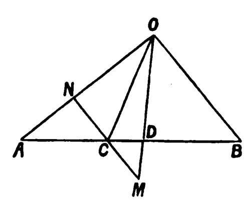
And, by MACCULLAGH's theorem,

$$TQ_1-TQ_2 = \text{arc } PQ_1-\text{arc } PQ_2.$$

Whence $TR-PR = TS-PS$, and, similarly, $TR'+P'R' = TS'+P'S'$.

13092. (Rev. G. H. HOPKINS, M.A.)—Obtain, by simple geometry, the harmonic mean between two given straight lines.

Solution by Rev. T. WIGGINS, B.A.; J. H. HOOKER; *and others.*



Let AB be the longer of the lines, and in it make AC = the shorter. Take any point O outside of AB and join OA, OC, OB. Through C draw MCN parallel to OB, making MC = CN. Join OM, cutting AB in D. Then AD is the required mean.

By similar Δs ACN and ABO, and MCD and OBD, we have

$$AC : AB = CN : BO,$$
$$DC : DB = CM : BO;$$
$$\therefore\ AC : AB = DC : DB \qquad (\because\ CM = CN),$$

which is the condition required that AD should be a harmonic mean between AC and AB.

13120. (C. E. BICKMORE, M.A.)—By some general process, express the prime 10,838,689 as the sum of two squares.

Solution by the PROPOSER, Professor MOREL, *and others.*

Let $10838689 = A$, and express $\sqrt{A}$ as a continued fraction.

The first three partial quotients are $3292, \quad 4, \quad 1$;

and the third complete quotient is $\dfrac{\sqrt{A}+1129}{2704}$;

$$\therefore \quad q_1 = 1, \quad q_2 = 4, \quad q_3 = 5, \quad \text{and} \quad p_3 = 1129q_3 + 2704q_2 = 16461.$$

But
$$p_3^2 - Aq_3^2 = -2704 = -52^2;$$

$$\therefore \quad A = \frac{p_3^2+52^2}{q_3^2} = \frac{16461^2+52^2}{3^2+4^2}$$

$$= \left(\frac{3\times 16461 - 4\times 52}{25}\right)^2 + \left(\frac{4\times 16461+3\times 52}{25}\right)^2$$

$$= 1967^2 + 2640^2.$$

13073. (Professor SANJÁNA.)—Construct a triangle geometrically, having given in length two sides and (1) the median between them, (2) the bisector of the angle between them, (3) the bisector of the angle exterior to them.

Solution by H. W. CURJEL, M.A.; Prof. RADHAKRISHNAN; *and others.*

(1) Construct $\triangle ABD$, having AD double the given median and the other sides equal to the given sides. Complete the parallelogram ABDC. Then ABC is clearly the required triangle.

(2) and (3) Let given sides $= a, b$, and bisector $= x$; find y, so that

$$y : x = a \pm b : a$$

(the upper sign being taken for Fig. 2, the lower for Fig. 3).

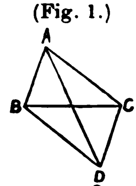
Make $\triangle CDA$, having

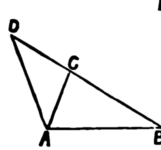
$$DA = y, \quad CD = CA = b;$$

produce DC (Fig. 2) or CD (Fig. 3) to B, making $CB = a$.

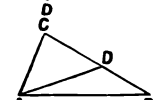
Then ABC is the required triangle.

(Fig. 1.)





(Fig. 2.)



(Fig. 3.)

13087. (H. J. WOODALL, A.R.C.S.)—If x is the least number for which $a(\text{Exp.}\, x)-1$ is divisible by y, find the least value of z for which $a(\text{Exp.}\, z)-1$ will certainly be divisible by y^2. (y prime to $a-1$.)

Solution by W. E. HEAL, M.A.; Professor LAMPE; *and others.*

Let $a^x-1 = ky$, $a^z-1 = k'y^2 = (k'y)\,y$; $z = tx$, a multiple of x.

$$a^z-1 = a^{tx}-1 = (a^x-1)\left\{a^{(t-1)x}+a^{(t-2)x}\ldots+a^{2x}+a^x+1\right\} \equiv 0 \mod y^2.$$

Let $k = rv$, $y = sv$, r and s prime to each other;

$$\therefore\quad a^x-1 \equiv 0 \mod sv^2,\quad a^z-1 \equiv 0 \mod s^2v^2;$$

$$\therefore\quad a^{(t-1)x}+a^{(t-2)x}\ldots+a^{2x}+a^x+1 \equiv 0 \mod s.$$

But $a \equiv 1 \mod s$; $\therefore$ $t = ws$, a multiple of s; $z = sx$.

[If k be prime to y, then $z = xy$.

Ex.—$2^5-1 = 31$; then $x = 155$ is the least value for which 2 (exp. x) -1 is divisible by $(31)^2$.]

13044. (Professor A. DROZ-FARNY.)—Représentons par Σ et Σ' les surfaces des deux triangles déterminés par les centres des carrés construits extérieurement ou intérieurement sur les côtés d'un triangle ABC; soit ω l'angle de BROCARD de ce triangle: on a $\cot\omega = 2\,(\Sigma-\Sigma')/(\Sigma+\Sigma')$.

Solution by Professors SANJÁNA, MUKHOPADHYAY, *and others.*

If the squares are all externally described, the area is easily seen to be $S+\frac{1}{8}(a^2+b^2+c^2)$; if all internally, it is $S-\frac{1}{8}(a^2+b^2+c^2)$. Hence

$$2\,(\Sigma-\Sigma')/(\Sigma+\Sigma') = 2\,.\,\tfrac{1}{4}(a^2+b^2+c^2)/2S = (a^2+b^2+c^2)/4S = \cot\omega.$$

13088. (D. BIDDLE.)—A series of improper fractions, of the form $A_n/(A_n-B_n)$, is such that $A_n = a^{2^{n-1}}$ and $B_n = B_{n-1}(A_{n-1}-B_{n-1})$. The first term is $a/(a-1)$. Prove that (1) the sum of n terms is

$$a^{2^n}/\left\{B_n\,(a^{2^{n-1}}-B_n)\right\}-a,\quad\text{or}\quad A_{n+1}/B_{n+1}-a,$$

and that (2) the continued product of the same terms is

$$a^{2^n}/\left\{a\,.\,B_n\,(a^{2^{n-1}}-B_n)\right\},\quad\text{or}\quad A_{n+1}/(a\,.\,B_{n+1}).$$

Also (3) give an easy formula for the immediate determination of B_n. [It is clear that, if a be prefixed (as a term) to the above series, the sum and the product will be identical.]

Solution by W. E. HEAL, M.A.; Professor SARKAR; *and others.*

(1) $$A_n/(A_n-B_n)+A_n/B^n = A_{n+1}/B_{n+1};$$

$$\therefore\quad \sum_{r=1}^{n} A_r/(A_r-B_r) = \sum_{r=1}^{n} A_{r+1}/B_{r+1}-\sum_{r=1}^{n} A_r/B_r = A_{n+1}/B_{n+1}-a.$$

(2) $$A_n/(A_n - B_n) = A_n B_n/B_{n+1};$$

$$\therefore \prod_{r=1}^{n} A_r/(A_r - B_r) = A_1 A_2 A_3 \ldots A_n/B_{n+1} = a^{(1+2+2^2+2^3\ldots+2^{n-1})}/B_{n+1}$$
$$= A_{n+1}/aB_{n+1}$$

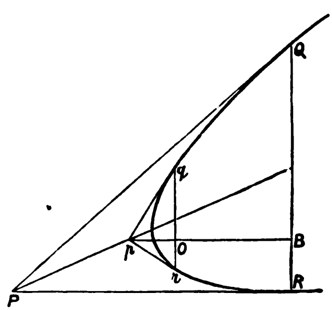
(3) $$A_n/2 - B_n = (A_{n-1}/2 - B_{n-1})^2 + (A_{n-1}/2)^2.$$

Whence, starting with $A_1 = a$, $B_1 = 1$, we can rapidly calculate $A_n/2 - B_n$.

[The PROPOSER cannot see the superiority of the method under (3) to that given in the question at the end of the second line.]

13102. (Professor COCHEZ.)—Une parabole tourne autour de son foyer. Aux points où elle rencontre une droite fixe, perpendiculaire à l'axe, on mène les tangentes à la courbe. Lieu des points d'intersection en ces tangentes.

Solution by H. W. CURJEL, M.A.; Prof. SWAMINATHA AIYAR; *and others.*



Let O be the fixed focus, and let the parabola whose axis makes an angle θ with the perpendicular OB on the fixed line QBR cut QBR in Q and R; let qr be the focal chord parallel to QR, and let P, p be the poles of QR, qr. Then angle $OpP = \pi - \theta$,

and $Pp = OB \sec\theta = b \sec\theta$,

and $Op = 2a \sec\theta$;

therefore, if x and y are the coordinates of P,

$x + b = -2a \sec\theta$, $y = -b \tan\theta$;

therefore locus of P is the hyperbola

$$\left(\frac{x+b}{2a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1.$$

9800. (W. J. C. SHARP, M.A.)—If a, b, A be given in a spherical triangle, deduce the conditions that the triangle should be impossible, unique, or ambiguous, from the discussion of the equation

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

where there are two triangles; show that, c and c' being the third sides,

$$\tan \tfrac{1}{2}(c + c') = \tan b \cos A,$$

and confirm this by the case when the radius of the sphere is infinite.

Solution by H. J. WOODALL, A.R.C.S.; Prof. GOPALACHANAR; *and others.*

Since $\cos a = \cos b \cos c + \sin b \sin c \cos A$, we get

$$\cos^2 c\,(\cos^2 b + \sin^2 b \cos^2 A) - 2\cos c \cos a \cos b + (\cos^2 a - \sin^2 b \cos^2 A) = 0;$$

whence $\cos c = \left\{\cos a \cos b \pm \sin b \cos A\,(\cos^2 b - \cos^2 a + \sin^2 b \cos^2 A)^{\frac{1}{2}}\right\} / (\cos^2 b + \sin^2 b \cos^2 A).$

Thus $\cos c$ will be impossible, unique, or ambiguous, according as

$$\cos^2 b + \sin^2 b \cos^2 A <, \; =, \text{ or } > \cos^2 a.$$

In the ambiguous case, we have

$$\cos a = \cos b \cos c + \sin b \sin c \cos A = \cos b \cos c' + \sin b \sin c' \cos A;$$

whence $\tan \frac{1}{2}(c+c') = (\cos c' - \cos c)/(\sin c - \sin c') = \tan b \cos A.$

If the radius becomes very large, we have

$$\tan \tfrac{1}{2}(c+c') = \tfrac{1}{2}(c+c') = \tan b \cos A = b \cos A;$$

$$\therefore \quad c + c' = 2b \cos A, \text{ which is obvious.}$$

3779. (Professor HUDSON, M.A.)—There are n problems of equal difficulty upon a paper, for which na minutes are allowed. A man who could do any one of them in na minutes tries for a minutes at each. If the chance of his doing any one be proportional to the time he tries at it, what fraction of the marks for the paper may he expect to get?

Solution by H. W. CURJEL, M.A.; Professor CHAKRIVARTI; *and others.*

The chance of his getting any one question right is $1/n$; therefore he may expect $1/n$ of the maximum marks for his paper.

12965. (M. BRIERLEY.)—Find a number which, if increased by a^2, the sum shall be a square; also, if one pth of it be added to a^2, the result shall be a square.

Solution by Professors RADHAKRISHNAN, CHAKRIVARTI, *and others.*

$$a^2 + x = y^2 \quad \text{and} \quad a^2 + x/p = z^2 \quad \ldots\ldots\ldots\ldots\ldots \; (1, 2);$$

whence $$y^2 - pz^2 = -(p-1)\,a^2.$$

Now, if $y = m$, $z = n$, be a solution of $y^2 - pz^2 = 1$, for which we may obtain an unlimited number of solutions,

$$y^2 - pz^2 = (a^2 - pa^2)(m^2 - pn^2) = a^2m^2 + p^2a^2n^2 - p\,(a^2m^2 + a^2n^2)$$
$$= (am \pm pan)^2 - p\,(am \pm an)^2.$$

Therefore $(am \pm pan)^2$ is a solution of y^2. Therefore the required number is $\quad a^2(m \pm pn)^2 - a^2$, by (1).

Thus, if $p = 6$, $m = 5$, $n = 2$ is one solution, $288a^2$ and $48a^2$ are numbers required; similarly $m = 49$, $n = 20$ will give another number, $28560a^2$, and so on.

13001. (Professor SANJÁNA.)—Solve the following equations:—

$$x^7+11x^6-12x^5-134x^4+428x^3-108x^2-432x+216 = 0;$$

$$x^8-197x^6+1260x^5-685x^4-8820x^3+13922x^2+1260x-2016 = 0.$$

Solution by G. HEPPEL, M.A.; H. W. CURJEL, M.A.; *and others.*

The equations may be written

$$(x+1)(x^2+4x-6)(x^2-4x+6)(x^2+10x-6) = 0,$$

$$(x-6)(x-8)(x^2+3x+1)(x^2-3x+1)(x^2+14x-42) = 0;$$

whence the solutions are obvious.

12987. (Rev. S. J. ROWTON, M.A.)—Is the following theorem, as given in text-books—that, when $n+1$ places of a square root have been obtained by the usual process, the next n places may be obtained by ordinary division only—true in every case? And, if not, where is the fallacy in the reasoning as generally given? If possible, give an example in which it does not hold, and explain why.

Solution by Professors SANJÁNA, RAMASWAMI AIYAR, *and others.*

(1) The rule, as worded above, is certainly inaccurate, as it does not take account of the figures in the root after the $(2n+1)$th place. TODHUNTER has given the correct enunciation by adding the words "supposing $2n+1$ to be the whole number" (of figures in the root). But the illustration he gives is inaccurate in principle, as $\sqrt{12}$ has an infinity of places in its evolution.

(2) Let the root have $2n+p+1$ figures *exactly*, of which $n+1$ are obtained by the ordinary method; to investigate if the next n could be obtained by ordinary division, let N be the number, $a+x+y$ the complete root, a the part found, x the part corresponding to the next n figures. Then $\sqrt{N} = a+x+y$ gives $\dfrac{N-a^2}{2a} = x+y+\dfrac{(x+y)^2}{2a}$.

Now $x+y > 10^{n+p-1}$ but $< 10^{n+p}$, and $2a > 2.10^{2n+p}$ but $< 2.10^{2n+p+1}$; hence $(x+y)^2/2a > \frac{1}{2}.10^{p-3}$ but $< \frac{1}{2}.10^{p}$. Thus we are not warranted in holding the remainder to be fractional; and y increased by this might have $p+1$ places, and so might affect the last digit of x when added on.

(3) *Illustration.*—$(12596358297783001)^{\frac{1}{2}} = 112233499$; if, after finding 1122 of the root, we proceed by ordinary division, we get 335. As the part neglected is 99 (y), and the remainder ($>\frac{1}{20}<50$) is in this case $(33499)^2/224400000$, *i e.*, actually a little greater than 5, we get 104.+ for the remainder after the division; and this, containing more than two (p) digits, changes the last digit in 334 (x). If we find 11223 first, and then proceed by division, we get the last four digits correct.

[Mr. BRILL remarks that this question has been fully discussed by Professor HILL in the *Proceedings* of the London Mathematical Society.]

9624. (Professor GRIESS.)—Soit Γ une ellipse donnée; sur le petit axe, comme diamètre, on décrit un cercle Δ. Par un point M, mobile sur Δ, on trace une tangente qui rencontre Γ aux points P, Q; soit M′ l'isotomique de M sur PQ (c'est-à-dire le symétrique de M par rapport au milieu de PQ). On demande le lieu décrit par M′. Ce lieu est une courbe unicursale du sixième ordre; on distinguera les différentes formes du lieu, suivant que l'on a $b > c$, ou $b < c$.

Solution by H. J. WOODALL, A.R.C.S.; Professor SARKAR; *and others.*

The chord of the ellipse joining the points (ϕ), (ϕ') is

$$x \cos \tfrac{1}{2}(\phi+\phi')/a + y \sin \tfrac{1}{2}(\phi+\phi')/b = \cos \tfrac{1}{2}(\phi-\phi');$$

if this coincides with $xx_1/b^2 + yy_1/b^2 = 1$ [the tangent to Δ at (x_1y_1)], we have $\quad b^2 \cos \frac{1}{2}(\phi+\phi')/ax_1 = b \sin \frac{1}{2}(\phi+\phi')/y_1 = \cos \frac{1}{2}(\phi-\phi')$, whence

$$b^2 = x_1^2 + y_1^2 = \left\{b^4 \cos^2 \tfrac{1}{2}(\phi+\phi') + a^2b^2 \sin^2 \tfrac{1}{2}(\phi+\phi')\right\}/a^2 \cos^2 \tfrac{1}{2}(\Phi-\phi');$$

$$\therefore \quad a^2 \cos^2 a'(\phi-\phi') = b^2 \cos^2 \tfrac{1}{2}(\phi+\phi') + a^2 \cos^2 \tfrac{1}{2}(\phi+\phi'),$$

which is the condition that the chord joining (ϕ), (ϕ') should be tangential to the given circle.

The mid-point (O) of the chord is $\left\{\frac{1}{2}a(\cos\phi + \cos\phi'),\ \frac{1}{2}b(\sin\phi+\sin\phi')\right\}$, the point sought (M′) is $\left\{a(\cos\phi+\cos\phi') - x_1,\ b(\sin\phi+\sin\phi') - y_1\right\}$.

The locus of O is $\ b^2x^2/a^2 + a^2y^2/b^2 = a^2(x^2/a^2 + y^2/b^2)^2$.

The locus of M′ can be found by eliminating $k \left\{= \cos^2 \frac{1}{2}(\phi-\phi')\right\}$ from the equations, and we have

$$(2a^2k - b^2)^2 (2bk - b)^2 = b^2x^2 (2bk-b)^2 + y^2 (2a^2k - b^2)^2.$$

13118. (J. J. BARNIVILLE, B.A.)—Prove that

$$\frac{1}{1.2} + \frac{1}{1.3} + \frac{1}{2.5} + \frac{1}{3.8} + \frac{1}{5.13} + \dots = 1;$$

$$\frac{1}{1.2} - \frac{1}{1.3} + \frac{1}{2.5} - \frac{1}{3.8} + \frac{1}{5.13} - \dots = \sqrt{5} - 2.$$

Solution by H. W. CURJEL, M.A.; Rev. T. ROACH, M.A.; *and others.*

$$\text{First series} = (1-\tfrac{1}{2}) + \left(\tfrac{1}{2} - \frac{1}{2.3}\right) + \left(\frac{1}{2.3} - \frac{1}{3.5}\right) + \left(\frac{1}{3.5} - \frac{1}{5.8}\right)$$
$$+ \left(\frac{1}{5.8} - \frac{1}{8.13}\right) + \dots = 1.$$

$$\text{Let } x = \frac{1}{1.2} + \frac{1}{2.5} + \frac{1}{5.13} + \dots = \frac{1}{2} + \frac{1}{2.3} - \frac{1}{3.5} + \frac{1}{5.8} - \frac{1}{8.13}$$
$$= \tfrac{1}{2} - \tfrac{1}{2} + \tfrac{2}{3} - \tfrac{2}{3} + \tfrac{3}{5} - \tfrac{3}{5} + \tfrac{5}{8} - \tfrac{5}{8} + \tfrac{8}{13} - \dots$$

($2n-1$ terms of this series corresponding to n terms of series just before),

$\frac{1}{1}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{5}{8}$, $\frac{8}{13}$, &c., being the successive convergents of the continued fraction $\quad \dfrac{1}{1+}\,\dfrac{1}{1+}\,\dfrac{1}{1+}\,\dfrac{1}{1+}\,\dfrac{1}{1+}\cdots;$

$$\therefore\quad x = \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\cdots = \frac{1}{1+x}; \qquad \therefore\quad x = \frac{\sqrt{5}-1}{2};$$

therefore second series $= 2x - S = \sqrt{5}-2$.

13122. (J. O'Byrne Croke, M.A.)—Find, by the use of a general theorem of relation, x, y, z from

$$x^2+y^2-z(x+y) = c^2, \quad y^2+z^2-x(y+z) = a^2, \quad z^2+x^2-y(z+x) = b^2.$$

Solution by R. F. Davis, M.A.; Prof. Mukhopadhyay; *and others.*

Putting $P = x+y+z$, $Q^2 = x^2+y^2+z^2$, the given equations may be transformed into the system $a^2 = Q^2-Px$, $b^2 = Q^2-Py$, $c^2 = Q^2-Pz$.

Adding, $\qquad a^2+b^2+c^2 = 3Q^2-P^2;$

squaring and adding, $\quad a^4+b^4+c^4 = 3Q^4-2P^2Q^2+P^2Q^2 = Q^2(3Q^2-P^2)$.

Hence $Q^2 = (a^4+b^4+c^4)/(a^2+b^2+c^2)$ and $P^2 = 3Q^2-(a^2+b^2+c^2) = $ &c.

Finally, $\quad x = \left\{b^4+c^4-a^2(b^2+c^2)\right\} / \left\{2(a^6+b^6+c^6-3a^2b^2c^2)\right\}^{\frac{1}{2}}$.

13091. (J. J. Barniville, B.A.)—Prove that, in the series $\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{8}+\frac{1}{13}+\ldots$, the sum of all the terms after the nth lies between u_{n-1} and u_n+u_{n+1}.

Solution by H. W. Curjel, M.A.; W. E. Heal, M.A.; *and others.*

Let S_n = sum of terms after the nth term, and $v_n = u_n^{-1}$;

then $\qquad v_{n+1} = v_n + v_{n-1};$

and $\quad v_{n-1}v_nv_{n+1}(u_{n+1}+u^n-u_{n-1}) = v_{n-1}v_n-(v_n-v_{n-1})\,v_{n+1}$

$$= -v_{n-2}v_{n-1}v_n(u_n+u_{n-1}-u_{n-2}) = (-1)^n A.$$

Putting $n = 2$, $\qquad A = 30\left(\frac{1}{3}+\frac{1}{5}-\frac{1}{2}\right) = 1;$

$$\therefore\quad u_{n+1}+u^n-u_{n-1} = (-1)^n u_{n-1}u_nu_{n+1},$$

$$u_{n+2}+u_{n+1}-u_n = (-1)^{n+1}u_nu_{n+1}u_{n+2}, \quad \text{\&c.}$$

Adding these results, $\quad S_n-u_{n-1} = (-1)^n$

(a positive quantity $< u_{n-1}u_{n+1}u_n$),

and $\qquad S_n-u_n-u_{n+1} = S_{n+1}-u_n = (-1)^{n+1}$

(a positive quantity $< u_nu_{n+1}u_{n+2}$)

therefore S_n lies between u_{n-1} and u_n+u_{n+1}.

13054. (Professor MORLEY.)—Prove that the locus of points whence two real tangents can be drawn to a helix is a system of helices.

Solution by H. W. CURJEL, M.A.; Prof. RADHAKRISHNAN; *and others.*

If a point be taken on any tangent to a helix at a distance r from the point of contact, then an equal tangent can be drawn from it, the points of contact being connected by the relation $\theta \sim \phi = 2/a = \tan\alpha$, say, where their coordinates are $(a\cos\theta,\ a\sin\theta,\ b\theta)$, $(a\cos\phi,\ a\sin\phi,\ b\phi)$; and the locus of the common point when r is constant is given by

$$x = (a^2+r^2)^{\frac{1}{2}}\cos(\theta+\alpha) = (a^2+r^2)^{\frac{1}{2}}\cos\tfrac{1}{2}(\theta+\phi),$$
$$y = (a^2+r^2)^{\frac{1}{2}}\sin(\theta+\alpha) = (a^2+r^2)^{\frac{1}{2}}\sin\tfrac{1}{2}(\theta+\phi),$$
$$z = b\tan\alpha + b\theta = b\left\{\tfrac{1}{2}(\theta+\phi)\right\},$$

and is therefore a helix of the same pitch.

4307. (Dr. ARTEMAS MARTIN.)—Find the average area of all the elliptical sections of a given right cylinder.

Solution by H. W. CURJEL, M.A.; Professor COONDOO; *and others.*

Let $2a\tan\alpha$ be the length of the cylinder, a being the radius of the base.

The average area $= \pi a^2 \int_0^\alpha \frac{\tan\alpha - \tan\theta}{\cos\theta}\,d\theta \Big/ \int_0^\alpha (\tan\alpha - \tan\theta)\,d\theta$

$$= \pi a^2 \Big[\tfrac{1}{2}\log\frac{1+\sin\theta}{1-\sin\theta} - \frac{1}{\cos\theta}\Big]_0^\alpha \Big/ (\alpha\tan\alpha + \log\cos\alpha)$$

$$= \pi a^2\left(\tfrac{1}{2}\log\frac{1+\sin\alpha}{1-\sin\alpha} - \frac{1}{\cos\alpha} + 1\right) \Big/ (\alpha\tan\alpha + \log\cos\alpha)$$

$$= \tfrac{1}{2}\pi a^2\left(\cos\alpha\log\frac{1+\sin\alpha}{1-\sin\alpha} + 2\cos\alpha - 2\right) \Big/ (\alpha\sin\alpha + \cos\alpha\log\cos\alpha).$$

[Mr. BIDDLE observes that, in order to allow of equal thickness for the elliptical laminæ whose areas are summed, the values to be integrated as given above should be multiplied, in both cases, by $\cos\theta$, giving the average area as

$$\pi a^2 \int_0^\alpha (\tan\alpha - \tan\theta)\,d\theta \Big/ \int_0^\alpha \cos\theta\,(\tan\alpha - \tan\theta)\,d\theta$$
$$= \pi a^2(\alpha\tan\alpha + \log\cos\alpha)/(\sec\alpha - 1).]$$

13071. (Professor CHAKRIVARTI.)—Find the area of a triangle from the radius (r) of the in-circle, the radius (r') of the circle described between the in-circle and the vertical angle, and the magnitude (2β) of

one of the base angles. Express the area in terms of r and r' when (1) the base angles are equal, (2) one of the base angles is right.

Solution by H. W. CURJEL, M.A.; Professor SARKAR; *and others.*

Let 2α be the vertical angle; then

$$\sin\alpha = (r-r')/(r+r'), \qquad \cos\alpha = \left\{2\,(rr')^{\frac{1}{2}}\right\}/(r+r').$$

$$\therefore\quad \text{area} = \tfrac{1}{2}r\,(\text{sum of sides})$$
$$= r^2\left[\cos\beta + \left\{2\,(rr')^{\frac{1}{2}}/(r+r')\right\} + \sin(\alpha+\beta)\right]$$
$$= 2r^2\left\{r\cos\beta + (rr')^{\frac{1}{2}}\,(1+\sin\beta)\right\}/(r+r').$$

(1) If the angles at the base are equal, $\beta = \frac{1}{4}\pi - \frac{1}{2}\alpha$.

Now
$$\cos\tfrac{1}{2}\alpha = \left\{(r)^{\frac{1}{2}} + (r')^{\frac{1}{2}}\right\}/\left\{2\,(r+r')\right\}^{\frac{1}{2}}.$$
$$\sin\tfrac{1}{2}\alpha = \left\{(r)^{\frac{1}{2}} - (r')^{\frac{1}{2}}\right\}/\left\{2\,(r+r')\right\}^{\frac{1}{2}};$$
$$\therefore\quad \sin\beta = (r')^{\frac{1}{2}}/(r+r')^{\frac{1}{2}}, \quad \cos\beta = (r)^{\frac{1}{2}}/(r+r')^{\frac{1}{2}};$$
$$\therefore\quad \text{area} = 2\,r^{\frac{3}{2}}\left\{(r+r')^{\frac{1}{2}} + (r')^{\frac{1}{2}}\right\}/(r+r').$$

(2) If $2\beta = \frac{1}{2}\pi$, area $= \sqrt{2}\,r^2\left\{r + (rr')^{\frac{1}{2}}(\sqrt{2}+1)\right\}/(r+r')$.

8966. (W. J. C. SHARP, M.A.)—If ρ, ρ_1, ρ_2, &c., be the radii of curvature at corresponding points on a curve, and its successive evolutes, then for a conic $\quad 45\rho\rho_1\rho_2 = 9\rho^2\rho_3 + 36\rho^2\rho_1 + 40\rho_1^3$,

and for a parabola $\quad 3\rho\rho_2 = 9\rho^2 + 4\rho_1^2$.

Solution by Rev. J. CULLEN, S.J.; Prof. KRISHMACHANDRA; *and others.*

Let $dy/dx = p$, $d^2y/dx^2 = q$, $\psi = \tan^{-1}p$, and $\sigma = (1+p^2)^{\frac{1}{2}}$; then we have $\rho = \sigma^3/q$, $\rho_1 = d\rho/d\psi$, $\rho_2 = d\rho_1/d\psi$, &c.

For the parabola and general conic (Vol. XXXVIII., p. 71),

$$(d/dx)^2\,(q^{-\frac{2}{3}}) = 0 \quad\text{and}\quad (d/dx)^3\,(q^{-\frac{2}{3}}) = 0 \quad \ldots\ldots\ldots\ldots \;(1,\,2).$$

But
$$\frac{d}{dx} = \frac{d\psi}{dx}\frac{d}{d\psi} = \frac{q}{\sigma^2}\frac{d}{d\psi} = \frac{\sigma}{\rho}\frac{d}{d\psi};$$

$\therefore$ (1) and (2) become $\left(\dfrac{\sigma}{\rho}\dfrac{d}{d\psi}\right)^2\left(\dfrac{\rho^{\frac{2}{3}}}{\sigma^2}\right) = 0$ and $\left(\dfrac{\sigma}{\rho}\dfrac{d}{d\psi}\right)^3\left(\dfrac{\rho^{\frac{2}{3}}}{\sigma^2}\right) = 0.$

Performing these operations, and remembering that

$$\frac{d\sigma}{d\psi} = \frac{dx}{d\psi}\frac{d\sigma}{dx} = p\sigma \quad\text{and}\quad \frac{dp}{d\psi} = \frac{dx}{d\psi}\frac{dp}{dx} = \sigma^2,$$

we obtain the above result.

APPENDIX.

UNSOLVED QUESTIONS.

4643. (Editor.)—Two points being taken at random on the perimeter of a rectangle, find the probability that an acute-angled triangle will be formed by joining these points with each other and with a third point taken at random (1) on the perimeter, (2) on the surface of the rectangle.

4645. (W. S. B. Woolhouse, F.R.A.S.) — A given straight line is divided at random into n parts, and these are arranged in the order of magnitude; find the average value of the part that stands the mth in order.

4650. (W. C. Otter.)—A spider at one end of a diameter of a circular pane of glass whose diameter is 30 inches gave uniform and direct chase to a fly moving uniformly along the circumference. The fly was 30° from the spider at the commencement of the chase, and the rate of the spider to that of the fly was as 3 : 2; find the nature of the curve traced by the spider in his pursuit, and the distance he will travel before he catches the fly.

4653. (Matthew Collins, LL.B.)—Let AC be the side of a regular pentagon inscribed in a circle whose centre is D; on AO draw an equilateral triangle AOB; bisect the arc BC in D, and the arc AD in E; then prove that the chord AE will be very nearly equal to the side of a regular undecagon inscribed in the circle.

4657. (J. W. L. Glaisher, F.R.S.)—An infinite number of perfectly flexible strings of the same length are knotted together at one end, and thrown down at random. Find the density at any distance from the knot.

4661. (Professor Cayley, F.R.S.)—It is required, by a real or imaginary linear transformation, to express the equation of a given cubic curve in the form $$xy-z^2 = \left\{(z^2-x^2)(z^2-k^2x^2)\right\}^{\frac{1}{2}}.$$

4662. (Prof. Salmon.)—Find (1) the degree of the evolute of a circular cubic or bi-circular quartic; and (2) how the characteristics of the evolute are affected when the curve is circular or bi-circular.

4663. (Professor Cremona.)—Les plans qui coupent en quatre points harmoniques une courbe gauche de 4ᵉ ordre et 2ᵉ espèce (courbe gauche unicursale de 4ᵉ ordre, sans points doubles) enveloppent une surface de Steiner pour laquelle la courbe donnée est asymptotique.

4665. (Professor Crofton, F.R.S.)—Four beams are jointed together, forming a parallelogram, the frame being kept in equilibrium by any four given forces at the joints, distortion being prevented by the stiffness of the joints. Prove that, if the bending moment on each joint be calculated as if it alone were to resist the distortion, the values will be the same for all four joints. What will the value be?

For *any* four-sided frame, at which joint would the greatest bending stress occur, if the remaining three joints are supposed free, as before?

4668. (Dr. Crum Brown.) — $4m+n$ separate strings, having each a black and a white end, are taken. $4m$ of these are united in groups of four by their white ends, and the white ends of all the others are to be left free. In how many ways may the black ends of the system be united two and two so as to form a continuous aggregate? It is, of course, contemplated that two black ends belonging to the same group of four may be united. This gives rise to the further question:—Divide the above arrangements into classes according to the number of groups of four, each of which has two of its black ends united.

4669. (F. A. Tarleton.)—Find the locus of the intersection of the perpendiculars of a triangle inscribed in one conic, and circumscribed about another.

4670. (S. Roberts.)—On a cubic curve there are in general twelve points at which nine successive points form an "ennead" through which a faisceau of cubics passes, and each curve of the faisceau has therefore a nine-point contact with the given curve. Let these be called "ennead points." Show that the ennead points may be parcelled out into four triads such that the equator of the curve referred to a triangle whose vertices form a triad consists of four terms. If the equation of the curve can be put in the form of three cubes, then one of the above forms also consists of three terms. The tangential of an ennead point is also an ennead.

4671. (Editor.)—If three points are taken at random on the surface of a sphere, and arcs of great circles drawn through each pair of them, find (1) the average area of all such triangles, and (2) the average area of their circumscribed circles.

4673. (S. Watson).—Through the angles of a given triangle straight lines are drawn at random, but so that a portion of each falls within the triangle. Find the average area of the triangle formed by their intersections.

4677. (T. Cotterill.)—Conicoids circumscribing a quartic curve in space have a common self-conjugate tetrahedron. Show that a plane cuts the quartic in four points, and the tetrahedron in four lines, such that triangles can be found circumscribing any conic inscribed in the four lines conjugate to any conic through the four points.

4680. (Prof. Burnside.) — (1) Determine the degree of the locus represented by the equation

$$\frac{\alpha^2}{p^2}+\frac{\beta^2}{p_1^{\,2}}+\frac{\gamma^2}{p_2^{\,2}}-2\frac{\beta\gamma}{p_1p_2}\cos A-2\frac{\gamma\alpha}{p_2p}\cos B-2\frac{\alpha\beta}{pp_1}\cos C=0,$$

where α, β, γ are the distances of any point from the vertices of the triangle ABC, and p, p_1, p_2 the perpendiculars of the same triangle.

(2) Show that the curve given by the above equation is the same as that referred to by Professor Cayley in Quest. 2110.

4682. (Prof. Evans.)—Find the quadrilateral of minimum perimeter that can be inscribed in a given rectangle so that one of its sides shall pass through a given point within the rectangle.

4864. (R. A. Roberts.)—Prove that the locus of points, whence tangents to a nodal cubic form an harmonic pencil, is a pair of harmonic cubics of which inflexions and inflexional tangents of the given cube are inflexions and inflexional tangents respectively.

4685. (W. S. B. Woolhouse, F.R.A.S.)—Four solids, *having any forms whatever*, being given, show how to determine sets of six points fixed in each, such that, if the four solids be placed in any positions in space and four points be taken arbitrarily, one in each solid, as the apices of a tetrahedron, the average square of volume in respect of all such variable tetrahedra shall always be equal to that of the 1296 tetrahedra formed by connecting all the groups obtainable from the four invariable sets of six points.

4698. (Rev. Dr. Booth, F.R.S.)—The difference between the quadrants of two ellipses whose semiaxes are a, b, and $(a+b)$, $2a^{\frac{1}{2}}b^{\frac{1}{2}}$, may be represented by a complete elliptic integral of the *first* order, or, in other words, it is equal to half the difference between the circumference of a spherical parabola and a semicircle, both described on a sphere whose radius is a.

4699. (Professor Crofton, F.R.S.)—A curve rolls on a straight line; determine the nature of the motion of one of its involutes.

4701. (W. S. B. Woolhouse, F.R.A.S.)—If, within a given closed area, three points be taken at random as the apices of a triangle, show that (1) the average of the square of the area of all such triangles will be reduced to one-third the value, if one of the points be fixed at the centre; also (2) if within a given volume of space four points be taken at random as the apices of a tetrahedron, the average of the square of the volume of all such tetrahedra will be reduced to one-fourth the value, if one point be fixed at the centre; and (3) that this theorem is true when the enclosed area or volume of space is *of any form whatever*.

4724. (Professor Clifford, F.R.S.)—If p, q be the foci; P, Q the asymptotes of a conic, θ the angle at a point a; and if at [A] there subtends the chord it cuts off from a line A: prove that (1) if a line B is drawn through the point a meeting the conic in l, m,

$$al \,.\, am \,.\, \sin \mathrm{BP} \sin \mathrm{BQ} = \frac{ap^2 \,.\, aq^2 \,.\, \sin^2\theta}{pq^2};$$

(2) if from a point b on the line A tangents L, M are drawn to the conic,

$$\sin \mathrm{AL} \sin \mathrm{AM} \,.\, bp \,.\, bq = \frac{\sin^2 \mathrm{AP} \sin^2 \mathrm{AQ} \,.\, [\mathrm{A}]^2}{\sin^2 \mathrm{PQ}};$$

where al means the distance between the points a, l; and BP means the angle between the lines B, P; also (3) find analogous propositions for a curve of any order on a plane or on a sphere.

4734. (Prof. Evans, M.A.)—Find the probability that

$$\cos\phi_1 + \cos\phi_2 + \cos\phi_3 > \sqrt{2},$$

where ϕ_1, ϕ_2, ϕ_3 are the angles of an acute-angled triangle.

4749. (A. F. Torry.)—From a point, at distance a from a centre of force varying as $r^{-2}+2ar^{-3}$, a particle is projected at an inclination of $\frac{1}{4}\pi$ to the initial distance. Determine the different orbits described, according as the initial velocity bears to that in a circle at the same distance a ratio greater than, equal to, or less than $2:\sqrt{3}$.

4757. (W. S. B. Woolhouse, F.R.A.S.)—Innumerable pairs of points are taken at random on the area of a given circle, their distance asunder not exceeding the radius of the circle; determine the law of the distribution of the collective mass of points.

4759. (Prof. Hudson, M.A.)—A right circular cylinder is made of elastic material attached to rigid fixed plane ends. It is distended by fluid pressure. Supposing that the tensions in the meridian and circular sections are regulated by Hooke's law, obtain equations sufficient to determine completely the shape it will assume. If the pressure p be constant, prove that the meridian curve is

$$x+A=\int\frac{\frac{1}{2}py^2+B}{\left\{(\lambda y^2/2a-\lambda y+C)^2-(\frac{1}{2}py^2+B)^2\right\}^{\frac{1}{2}}}\,dy.$$

4760. (Prof. Burnside, M.A.)—A binary quantic of the $2n$th degree in x, y may be considered as a ternary quantic of the nth degree in y^2, x^2, $2xy$, having $\frac{1}{2}n(n-1)$ less than the proper number of constants. Prove that, in virtue of this reduction in the number of independent constants, it is possible to arrange the terms so that any invariant or covariant of the ternary form is an invariant or covariant of the binary form; and, moreover, that the additional invariants and covariants of the $2n$-ic are invariants and covariants of the conjoint system, composed of the ternary form of the nth degree, with a ternary quadric.

4763. (Prof. Genese, M.A.)—An ellipse turns about its centre; find the envelope of the chords of intersection with the initial position. Also, if the ellipse move parallel to its major axis, find the envelope of the chords of intersection with the initial position of the axes.

4790. (R. Tucker, M.A.)—Given the inscribed circle of a triangle, and its point of contact with the base; find the locus of the vertex when (1) the perimeter, (2) the sum of the sides, (3) the difference of the sides, (4) the product of the sides is constant. Find a locus of the centre of the circumscribing circle, under the same conditions.

4792. (Prof. Sylvester, F.R.S.)—Let $\lambda_1, \lambda_2 \dots \lambda^n$ be any n positive quantities whatever, and let

$$C_1=\lambda_1,\quad C_2=1/\lambda_1+\lambda_2,\ \dots\ C_{n-1}=1/\lambda_{n-2}+\lambda_{n-1}\,C_n=1/\lambda_{n-1}.$$

Also, let $X_1, X_2, X_3, \dots X_n$ be any n linear fractions of x, and Fx the denominator of $\frac{1}{X_1-}\,\frac{1}{X_2-}\,\frac{1}{X_3}-\frac{1}{X_n}$, and make $X_1=c$, $X_2=c_2\dots X_n={}_nc$. Then, if these n equations are satisfied by the same values of x, this will be a root of Fx; but, if this is not the case, the greatest and least values of x derived from the above equations will be respectively superior and inferior limits to the roots of Fx.

[From the above it will be seen that the *quotients* in Sturm's theorem may be utilized to obtain *superior and inferior limits* to the roots of an equation.]

4800. (C. B. S. Cavallin.)—Find the average area of the triangle cut off by a random line from an equilateral triangle.

4803. (Dr. Collins.)—If A, B, C, D, E be five points in space, and if a = vol. of pyramid BCDE, b = vol. of pyramid ACDE, &c.; prove that the condition that one of the five points should be within the pyramid formed by the other four is

$$\tfrac{1}{2}S_1S_4 = S_2S_3 - S_5 + 4ABCDE, \quad \text{where } S_n = a^n + b^n + c^n + d^n + e^n.$$

4812. (G. A. Ogilvie.)—In how many cases is it impossible to succeed in getting two packs of 10 cards (numbered from 1 to 10) arranged in order, (1) being allowed to make two packs besides those with which we terminate, (2) being allowed three extra packs? Extend this to the case of an ordinary pack of cards.

4813. (I. H. Turrell.)—Construct a triangle geometrically, having given the vertical angle, radius of inscribed circle, and the centre of gravity of the triangle on the circumference of the inscribed circle.

4815. (Dr. Artemas Martin.)—If four dice be piled up at random on a horizontal plane, find the probability that the pile will not fall down.

4816. (Prof. Evans, M.A.)—A. hits a circular target of twenty inches radius, at the distance of four hundred yards, five hundred times out of one thousand shots with a rifle. How many times out of one thousand shots with the rifle must B. hit the same target, at the same distance, to show that his skill is to the skill of A. as two to one?

4824. (Professor Wolstenholme.)—A large area is to be paved with circular discs of black marble which are to touch; the interstices to be filled up with slabs of white marble. Each piece of pavement must be cut from a square slab: if the price of the slabs per foot of area vary as the length of a side, and if the cost of cutting the marble be a shillings per linear foot, and the price of a slab one foot square be b shillings, show that, in order that the area may be paved with least expense, the radius of each disc must be $(2\pi a^{-1})^{\frac{1}{2}}$.

4834. (Prof. Genese, M.A.)—If a ball be impelled from a fixed point on a smooth billiard table against a rough cushion, it will, after impact, appear to proceed from a fixed point behind the cushion. Hence also show that a ball played from baulk spot may return into baulk without the use of "side."

4841. (Prof. Sylvester, F.R.S.)—I happened to know the time required to walk from one railway to another by two roads at right angles to each other. I walked at the rate of 3 miles per hour, and recently applied this knowledge to determine the probable fare in driving from the one station to the other in a direct line. It is easy to see that, if the sum of the distances by the two rectangular routes is δ, the average direct distance is $4/(2+\pi)\,\delta$, or about one-fifth less. This suggested to me the further question, which I submit to the readers of the *Educational Times*. If a person walks from one point to another so as always to maintain the same distance from the middle point between them, and never to be retrogressing, the distance travelled will evidently be to the direct distance as $\pi : 2$. Now, suppose that the journey is accomplished by an unknown number of changes of direction, subject to the condition that the traveller is never

to increase his distance from the middle point between the two termini nor from the final terminus. The average distance travelled will evidently bear to the shortest distance some ratio intermediate between 1 and $\frac{1}{2}\pi$. Required its value.

4842. (Professor Cayley, F.R.S.) — *Trace* the curve defined by the equations given in the solution of Question 2110. [Vol. VII., for January 1867, pp. 17–19.]

4843. (Professor Clifford, F.R.S.)—If, in regard to a system of n quadric surfaces, the two systems of n polar planes in regard to any two points of space are projective to one another, either the quadrics have a common Jacobian or each of them is a doubled plane.

4845. (Professor Crofton, F.R.S.) — A line is drawn through two points taken at random inside a given triangle; find the mean area of the *triangular* portion cut off from the given triangle.

4847. (Sir R. S. Ball.)—A rigid body capable of rotating around a fixed point, is in stable equilibrium. If the body, when slightly displaced from its position by being rotated around an axis, continues for ever to vibrate around this axis, this line is called a normal axis. Prove that there are in general three normal axes; that, when forces have a potential, the three normal axes are conjugate diameters of the momental ellipsoid, and that they may be completely determined by a geometrical construction.

4864. (C. B. S. Cavallin.)—Two points P and Q are taken at random on the area of a vertical circle; find the probability that the time of descent for a particle down the straight line PQ, from P to Q, may be less than that from P down the straight line of quickest descent to the circle.

4876. (S. Tebay, B.A.)—ABCD is a quadrilateral, and O the intersection of the diagonals; *abcd* is a quadrilateral formed at random, having its angles in AOB, BOC, COD, DOA. Find the probability that it is reentrant.

4881. (Dr. Hart.) — Describe a conic which shall pass through the four vertices of a given parallelogram, and touch a given conic concentric with the parallelogram.

4884. (C. B. S. Cavallin.)—Three straight lines are drawn at random across a triangle; show that the probability that each line cuts unequal pairs of sides is $16\,(a+b+c)^{-4}\Delta^2$, where Δ is the area of the triangle, and a, b, c its sides.

4886. (G. A. Ogilvie.)—Find the conditions in order that an equation of the $2m$th degree may represent m concentric touching conics.

4891. (C. W. Merrifield, F.R.S.)—Give a general method of quadratures for finding $\int_a^b y\,dx$ from $\frac{dy}{dx}(=p) = f(x)$, where both y and p become infinite at one of the limits.

[An example would be to find $\int_0^1 x\,dp$ from $\frac{dp}{dx} = (4-3p-p^3)^{\frac{1}{2}}$, where x, y, and p all vanish together.]

4893. (Dr. Artemas Martin.)—Three points are taken at random on the surface of a hemisphere and joined by arcs of great circles; find the chance that the area of the spherical triangle thus formed is less than one-fourth of the surface of the hemisphere.

4896. (Prof. Sylvester, F.R.S.)—Show that, if a circle can be drawn touching the directions of the connecting rod, the radii, and the line of centres in a 3-link-work, then, and then only, is the motion of every point rigidly attached to the connecting rod unicursal.

4903. (Editor.)—Divide unity into four parts such that, if the square of one of them be diminished by four times the product of the other three, the remainder may be a rational square; and extend the problem to other cases.

4909. (Prof. Genese, M.A.)—Find functions f and ϕ of two variables, so that $\alpha = f(xy)$, $\beta = \phi(xy)$, $x = f(\alpha\beta)$, $y = \phi(\alpha\beta)$.

4910. (S. A. Renshaw.)—Round any point in the plane of a conic from which a pair of tangents can be drawn to the conic, a circle may be drawn such that, if tangents be drawn to it from the focus (F) of the conic, the intersections (m, n) of the two pairs of tangents to the conic and circle lie in a straight line passing through the point (Z), in which the chord of contact of tangents to the conic meets the directrix; and furthermore, if FZ, SZ be joined (S being the centre of the circle), then in the line mn there exists a point (O) such that, if through it a parallel be drawn to FZ and meeting the tangents to the circle in p and q, and also through O a parallel be drawn to SZ, and meeting the tangents to the conic in r and s; the figure $prqs$ will be a parallelogram.

4911. (S. Watson.)—Three points are taken at random, one on each side of a given triangle; find the average area of the circle drawn through them.

4912. (Dr. Artemas Martin.)—A cube is cut at random by a plane; find the respective probabilities that the plane cuts three, four, five, or all of the faces of the cube.

4916. (S. Tebay, B.A.) — Shocking young lady, indeed! "Oh, Charles, isn't it fun? I've beaten Arthur and Julia, and I've broke Aunt Sally's nose seven times." (*Punch*, June 16, 1860.) Given the velocity and the angular velocity of the stick, compare the probabilities of breaking Aunt Sally's nose when the centre of the stick passes to the right and to the left of Aunt Sally's head.

4919. (C. H. Hinton.)—A cylindrical cask, a feet deep and $2r$ feet in diameter, is full of wine. Water can be let in at the top at the rate of b gallons per minute, and there is a pipe in the centre of the bottom, c inches in diameter, through which, when open, the mixture can escape. If the supply and discharge pipes be both opened at the same instant, how much *wine* will remain in the cask at the end of t minutes, supposing the two fluids to mingle perfectly?

4923. (Professor Cayley, F.R.S.) — Find the value of the elliptic integral F (c, θ) when c is very nearly $= 1$ and θ very nearly $= \frac{1}{2}\pi$; that

is, the value of $\int_0^{\frac{1}{2}\pi-a} \frac{d\theta}{\{1-(1-b^2)\sin^2\theta\}}$, where a, b are each of them indefinitely small.

[The Proposer remarks that, for $a=0$, b small, the value is $=\log 4/b$, and for $b=0$, a small, the value is $-\log\cot\frac{1}{2}a$.]

4927. (Sir R. S. Ball.)—If k be the constant term in the equation of a surface, and $\Delta=0$, the condition necessary that this surface and three others pass through a point, what is the geometrical meaning of the roots of the equation $e^{-x(d/dk)}\Delta=0$?

4938. (Prof. Burnside, M.A.)—Find the locus of points such that the polar conics with reference to the curve U shall be equilateral hyperbolas, where $U=\Sigma\frac{\sin 2A}{xyz+x^2(-x\cos A+y\cos B+z\cos C)}$; and show that this locus passes through the vertices of the triangle ABC, and through the feet of the perpendiculars of the same triangle.

4943. (Matthew Collins.)—Subtract the sum of every two of the five numbers A, B, C, D, E from the sum of the remaining three of them, and we thus obtain ten different remainders (one of which is $A+B+C-D-E$); find the symmetric function which is the continued product of these ten remainders; and, if $aA^8B^2+bA^8BC+cA^7B^3+dA^7B^2C+eA^7BCD+fA^6BCDE+gA^2B^2C^2D^2E^2+$ &c. be a few terms of the said product, prove that $a=-3$, $b=6$, $c=8$, $d=0$, $e=-16$, and *especially* find f and g.

4944. (Rev. J. Blissard, B.A.)—Required to show that, if

$$f(x)=\frac{c_1x}{1^2}-\frac{c_2x(x-1)}{1.2^2}+\frac{c_3x(x-1)(x-2)}{1.2\cdot 3^2}-\text{\&c.},$$

then c_1, c_2, c_3, ..., can be so determined that

$$\frac{1}{1}+\frac{1}{m+1}+\frac{1}{2m+1}\ \text{\&c.}+\frac{1}{(n-1)m+1}$$

may be equated to any one of the m functions, viz., $f(mn)$, $f(mn-1)$, $f(mn-2)\ldots f(mn-m+1)$, which functions, therefore, are all of equal value.

4945. (Prof. Sylvester, F.R.S.) — If $\rho=a+b\cos\theta$, $\rho=b+a\cos\theta$ represent two limaçons, one of them will be entirely closed within the other, and the two will touch each other internally at the point in the axis remotest from the origin. Now, let either of them revolve about the common tangent until they come again into the same plane, they will then touch each other externally. Prove that, when one of these limaçons touching each other externally is fixed, and the other rolls upon it, each of its points will describe the inverse of a nodal cubic.

4950. (Professor Clifford, F.R.S.)—Prove that every matrix of the second order may be expressed in the form $a\mathrm{I}+b\mathrm{J}$, where I is the matrix unity and J a matrix such that $\mathrm{J}^2=-\mathrm{I}$. Hence find an expression for any power of such a matrix. (See Cayley on Matrices, *Phil. Trans.*, 1858.) Required a geometrical representation for a non-self-conjugate linear and vector function.

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